Extending Harrison's induction map to non-abelian groups

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Contents

1. Galois extensions

Throughout this talk, K is a finite extension of $\mathbb{Q}$.

The notion of a Galois extension of K is due to M. Auslander and O. Goldman [AG59].

Let A be a finite dimensional commutative K-algebra and let $\mathrm{End}_K(A)$ denote the K-algebra of K-linear maps $\phi : A \to A$. Let $\mathrm{Aut}_K(A)$ denote the group of K-algebra automorphisms of A.

Let F be a finite subgroup of $\mathrm{Aut}_K(A)$ with $K = A^F$. Let $D(A, F)$ denote the *crossed product algebra* of A by F.

Then A is an *F-Galois extension of K* if the map

$$j : D(A, F) \to \mathrm{End}_K(A),$$

defined as $j(\sum_{g\in F} a_g g)(t) = \sum_{g\in F} a_g g(t)$, $a_g, t \in A$, is an isomorphism of K-algebras.

The notion of F-Galois extension generalizes the usual definition of a Galois extension of fields.

Let A, A' be F-Galois extensions of K. Then *A is isomorphic to A' as F-Galois extensions of K* if there exists an isomorphism of commutative K-algebras $\theta : A \to A'$ for which

$$\theta(g(x)) = g(\theta(x))$$

for all $g \in F$, $x \in A$.

Let $\mathcal{G}al(K, F)$ denote the set of isomorphism classes of F-Galois extensions of K.

Example 1.

Let $\mathrm{Map}(F, K)$ denote the K-algebra of maps $\phi : F \to K$. Then $\mathrm{Map}(F, K)$ is the *trivial F-Galois extension* of K with action defined as

$$g(\phi)(h) = \phi(g^{-1}h)$$

for $g, h \in F$, $\phi \in \mathrm{Map}(F, K)$.

The set of maps $\{v_g\}_{g \in F}$ defined as

$$v_g(h) = \delta_{g,h},$$

for $g, h \in F$, is a K-basis for $\mathrm{Map}(F, K)$.

$\square$

The Galois extensions are completely determined by the following.

Theorem 2.

Let K be a field, let F be a finite group and let A be an F-Galois extension of K. Then

$$A = \underbrace{L \times L \times \cdots \times L}_{n}$$

where L is a U-Galois field extension of K for some subgroup U of F of index n. (L is a Galois extension of K with group U in the usual sense.)

Proof.

See[Pa90, Theorem 4.2]. $\square$

One of the main points of this talk is to answer the following question: Can we recover A from the data U, L? In other words: can A be induced from the data U, L?

The answer is yes (of course), but the method of doing so seems to depend on whether F is abelian or not.

2. Harrison's induction map in the abelian case

If F is abelian, then Harrison has given an induction map, which we now describe.

Let $U \leq F$ and suppose that $M \in \mathcal{G}al(K, U)$.

Then by Auslander and Goldman [AG60],

$$M \otimes_K \mathrm{Map}(F, K)$$

is a $(U \times F)$-Galois extension of K.

The Galois action of $(U \times F)$ on $M \otimes_K \mathrm{Map}(F, K)$ is defined as follows: for $m \otimes a \in M \otimes_K \mathrm{Map}(F, K)$,

$$(u, g)(m \otimes a) = u(m) \otimes g(a).$$

Since F is abelian, there is a homomorphism of groups

$$\psi : U \times F \to F,\ (u, g) \mapsto ug,$$

with $(U \times F)/\ker(\psi) \cong F$.

By Chase, Harrison, Rosenberg [CHR65],

$$(M \otimes_K \mathrm{Map}(F, K))^{\ker(\psi)}$$

is an F-Galois extension of K.

In this way, we define a map

$$T_U : \mathcal{G}al(K, U) \to \mathcal{G}al(K, F),$$

$M \mapsto (M \otimes_K \mathrm{Map}(F, K))^{\ker(\psi)}$, which is the *Harrison induction map* [Ha65].

The main result of Harrison [Ha65] is the following.

Theorem 3 (Harrison).

Let F be a finite abelian group. Let A be an F-Galois extension of K. Then there exists a subgroup $U \leq F$ and a U-Galois extension L of K for which

$$T_U(L) = A,$$

i.e., A can be induced from the data U, L.

A sketch of the proof is worth looking at.

Proof.
By Theorem 2,

$$A = \underbrace{L \times L \times \cdots \times L}_{n}, \tag{1}$$

where L is a U-Galois field extension of K for some subgroup U of F of index n.

Actually, if ϵ is the minimal idempotent corresponding to the first component of (1), then $U = \{g \in F : \ g\epsilon = \epsilon\}$ and $L = A\epsilon$.

Since $U \leq F$, there is a Harrison induction map:

$$T_U : \mathcal{G}al(K, U) \to \mathcal{G}al(K, F),$$

defined as follows: for $M \in \mathcal{G}al(K, U)$,

$$T_U(M) = (M \otimes_K \mathrm{Map}(F, K))^{\ker(\psi)}.$$

We need to show that

$$A \cong T_U(L) = (L \otimes_K \mathrm{Map}(F, K))^{\ker(\psi)},$$

as F-Galois extensions of K.

To this end, we define a map

$$\theta : A \to (L \otimes_K \mathrm{Map}(F, K))^{\ker(\psi)}$$

by the rule

$$\theta(a) = \sum_{g \in F} g^{-1}(a)\epsilon \otimes v_g,$$

for $a \in A$.

Then, θ is an isomorphism of F-Galois extensions of K.

Indeed, let $f \in F$ (f is identified with $(1, f) \in U \times F$). Then

$$\begin{aligned} f(\theta(a)) &= (1, f)(\theta(a)) \\ &= (1, f) \sum_{g \in F} g^{-1}(a)\epsilon \otimes v_g \\ &= \sum_{g \in F} g^{-1}(a)\epsilon \otimes v_{fg}. \end{aligned}$$

Now, replacing g with $f^{-1}g$ yields

$$\begin{aligned} \sum_{g \in F} g^{-1}(a)\epsilon \otimes v_{fg} &= \sum_{f^{-1}g \in F} (f^{-1}g)^{-1}(a)\epsilon \otimes v_{f(f^{-1}g)} \\ &= \sum_{f^{-1}g \in F} g^{-1}(f(a))\epsilon \otimes v_g \\ &= \theta(f(a)). \end{aligned}$$

$\square$

The main point is that in the case that F is abelian, every F-Galois extension of K can be written in the form

$$A = (L \otimes_K \mathrm{Map}(F, K))^{\mathsf{ker}(\psi)},$$

for some subgroup $U \leq F$ and some $L \in \mathcal{G}al(K, U)$.

Or: for F abelian, a given F-Galois extension A gives rise to the data U, L and this data can be used to recover the original F-Galois extension A.

3. The Induction map in the general case

In the case that F is non-abelian, the construction of the map T_U as above is not possible: the map $U \times F \to F$ may not be a homomorphism.

Pareigis [Pa90], has addressed this issue and has extended the map T_U (now written $\mathcal{T}_U$) to include non-abelian groups.

Moreover, the analog of Harrison's main result holds: an arbitrary F-Galois extension A can be induced from some pair U, L, i.e., $\mathcal{T}_U(L) = A$ for some $U \leq F$ and some U-Galois extension L/K.

We describe Pareigis' induction map.

Let $U \leq F$ and suppose that $M \in \mathcal{G}al(K, U)$ with $n = [F : U]$.

Let $T = \{g_1, g_2, \ldots, g_n\}$ be a left transversal for U in F and let

$$A = \underbrace{M \times M \times \cdots \times M}_{n}$$

with minimal orthogonal idempotents $e_1, e_2, \ldots, e_n$.

Let $\varsigma : F \to S_n$ be defined as $\varsigma(f)(i) = j$ if $fg_iU = g_jU$.

Define an action of F on A as follows: for $f \in F$,

$$f(\sum_{i=1}^{n} m_i e_i) = \sum_{i=1}^{n} (g_{\varsigma(f)(i)}^{-1} f g_i)(m_i) e_{\varsigma(f)(i)},$$

for $m_i \in M$, $1 \leq i \leq n$.

Then A is an F-Galois extension of K [Pa90, Theorem 4.2].

In this way, we define a map

$$\mathcal{T}_U : \mathcal{G}al(K, U) \to \mathcal{G}al(K, F),$$

$M \mapsto \underbrace{M \times M \times \cdots \times M}_{n}$, which is the *Pareigis induction map*.

We can now extend Harrison's result to any finite group.

Theorem 4.

Let F be a finite group. Let A be an F-Galois extension of K. Then there exists a subgroup $U \leq F$ and a U-Galois extension L of K for which

$$\mathcal{T}_U(L) = A,$$

i.e., A can be induced from the data U, L.

Proof. (Sketch.)
By Theorem 2,

$$A = \underbrace{L \times L \times \cdots \times L}_{n}, \tag{2}$$

where L is a U-Galois field extension of K for some subgroup U of F of index n.

Actually, if ϵ is the minimal idempotent corresponding to the first component of (2), then $U = \{g \in F : \ g\epsilon = \epsilon\}$.

Now with the data U, L, we construct the F-Galois extension A' using the induction map of Pareigis:

$$A' = \mathcal{T}_U(L) = \underbrace{L \times L \times \cdots \times L}_{n},$$

using the left transversal $\{g_1, g_2, \ldots, g_n\}$ for U in F.

We claim that $A \cong A'$ as F-Galois extensions of K. To this end we define a map

$$\theta : A \to A'$$

by the rule

$$\theta(\sum_{i=1}^{n} l_i e_i) = \sum_{i=1}^{n} g_i^{-1}(l_i) e_i,$$

for $l_i \in L$, $1 \leq i \leq n$.

Now, for $f \in F$,

$$
\begin{aligned}
f(\theta(\sum_{i=1}^{n} l_i e_i)) &= f(\sum_{i=1}^{n} g_i^{-1}(l_i)e_i) \\
&= \sum_{i=1}^{n}(g_j^{-1} f g_i)(g_i^{-1}(l_i))e_j, \quad j = \varsigma(f)(i) \\
&= \sum_{i=1}^{n} g_j^{-1}(f(l_i))e_j, \quad j = \varsigma(f)(i) \\
&= \theta(\sum_{i=1}^{n} f(l_i)e_j), \quad j = \varsigma(f)(i) \\
&= \theta(f(\sum_{i=1}^{n} l_i e_i)).
\end{aligned}
$$

And so, $\theta : A \to A'$ is an isomorphism of F-Galois extensions.

$\square$

Consequently, for any finite group F, every F-Galois extension of K can be written in the form

$$A = \mathcal{T}_U(L),$$

for some subgroup $U \leq F$ and some element $L \in \mathcal{G}al(K, U)$, where $\mathcal{T}_U$ is the Pareigis induction map.

Equivalently, a given F-Galois extension A gives rise to the data U, L and this data can be used to recover the original F-Galois extension A.

So, given a finite group F, a subgroup $U \leq F$, and a U-Galois extension L/K, we can construct (induce) an F-Galois extension A using the Pareigis induction map:

$$A = \mathcal{T}_U(L).$$

In the case that F is abelian, we can construct another F-Galois extension B using the Harrison induction map:

$$B = T_U(L).$$

And we expect (of course!) that $A \cong B$ as F-Galois extensions of K. Here is an example.

Example 5.

Let $F = C_4 = \{1, \sigma, \sigma^2, \sigma^3\}$ be the cyclic group of order 4 and let $U = C_2 = \{1, \sigma^2\}$ denote the cyclic subgroup of order 2. Let $L = \mathbb{Q}(\sqrt{2})$, so that C_2 is the Galois group of $L/\mathbb{Q}$.

If we use the Pareigis induction map with the data $C_2, \mathbb{Q}(\sqrt{2})$, we obtain the C_4-Galois extension

$$A = \mathbb{Q}(\sqrt{2})e_1 \oplus \mathbb{Q}(\sqrt{2})e_2,$$

where the C_4-Galois action is defined by

$$\sigma((a_0 + a_1\sqrt{2})e_1 \oplus (b_0 + b_1\sqrt{2})e_2) = (b_0 - b_1\sqrt{2})e_1 \oplus (a_0 + a_1\sqrt{2})e_2,$$

for $a_0, a_1, b_0, b_1 \in \mathbb{Q}$.

Now, using the Harrison induction map, the same data $C_2, \mathbb{Q}(\sqrt{2})$, yields the C_4-Galois extension

$$B = (\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathrm{Map}(C_4, \mathbb{Q}))^{\mathsf{ker}(\psi)}.$$

where

$$\mathsf{ker}(\psi) = \{(1,1), (\sigma^2, \sigma^2)\}$$

is the kernel of the group homomorphism $\psi : C_2 \times C_4 \to C_4$, defined by group product.

By direct computation of the fixed ring, every element of B has the form

$$(a_0 + a_1\sqrt{2}) \otimes v_1 + (a_0 - a_1\sqrt{2}) \otimes v_{\sigma^2}$$
$$+(b_0 + b_1\sqrt{2}) \otimes v_\sigma + (b_0 - b_1\sqrt{2}) \otimes v_{\sigma^3},$$

for some $a_0, a_1, b_0, b_1 \in \mathbb{Q}$.

The C_4-Galois action is given as

$$\begin{aligned}(1,\sigma)((a_0 + a_1\sqrt{2}) &\otimes v_1 + (a_0 - a_1\sqrt{2}) \otimes v_{\sigma^2} \\ &+(b_0 + b_1\sqrt{2}) \otimes v_\sigma + (b_0 - b_1\sqrt{2}) \otimes v_{\sigma^3}.) \\ = (b_0 - b_1\sqrt{2}) &\otimes v_1 + (b_0 + b_1\sqrt{2}) \otimes v_{\sigma^2} \\ &+(a_0 + a_1\sqrt{2}) \otimes v_\sigma + (a_0 - a_1\sqrt{2}) \otimes v_{\sigma^3}).\end{aligned}$$

Evidentially, $A \cong B$ as C_4-Galois extensions of K.

□

4. The Haggenmüller-Pareigis bijection

Now, let N denote a finitely generated group with finite automorphism group $F = \mathrm{Aut}(N)$.

Let $K[N]$ denote the group ring K-Hopf algebra and let B be a finite dimensional commutative K-algebra.

A *B-form* of $K[N]$ is a K-Hopf algebra A for which

$$B \otimes_K A \cong B \otimes_K K[N] \cong B[N]$$

as B-Hopf algebras.

A *form* of $K[N]$ is a K-Hopf algebra for which there exists a commutative, finite dimensional K-algebra B with

$$B \otimes_K A \cong B \otimes_K K[N] \cong B[N],$$

as B-Hopf algebras.

The *trivial form* of $K[N]$ is $K[N]$.

Let $\mathcal{F}orm(B/K, K[N])$ denote the collection of the isomorphism classes of the B-forms of $K[N]$ and let $\mathcal{F}orm(K[N])$ denote the collection of the isomorphism classes of the forms of $K[N]$.

Due to R. Haggenmüller and B. Pareigis [HP86, Theorem 5], there exists a bijection

$$\Theta : \mathcal{G}al(K, F) \to \mathcal{F}orm(K[N]),$$

which gives a 1-1 correspondence between the isomorphism classes of F-Galois extensions of K and forms of $K[N]$, where $F = \mathrm{Aut}(N)$.

For an F-Galois extension A of K, the map Θ is given explicitly as the fixed ring

$$\Theta(A) = (A[N])^F,$$

where F acts on A through the Galois action and on N as automorphisms. The fixed ring $(A[N])^F$ is an A-form of $K[N]$ and so belongs to $\mathcal{F}orm(K[N])$.

When $N = \mathbb{Z}$, C_3, C_4, or C_6, we have $F = \mathrm{Aut}(N) = C_2$.

In these cases, Θ has been used to construct all of the Hopf algebra forms of the group ring Hopf algebra $K[N]$ (they are the images of the quadratic extensions of K). See [HP86, Theorem 6].

In the paper [KU25], Kohl and U. have computed the preimages under Θ of certain Hopf forms of $K[N]$. These Hopf forms arise in the following way.

Let E/K be a Galois extension with group G, and let $\lambda : G \to \mathrm{Perm}(G)$ denote the left regular representation.

Let $(H_N, \cdot_N)$ be a Hopf-Galois structure on E/K corresponding to a regular subgroup $N \leq \mathrm{Perm}(G)$ that is normalized by $\lambda(G)$.

Necessarily, $|N| = |G|$, yet we may have $N \ncong G$ as groups; N is the *type* of $(H_N, \cdot_N)$.

Moreover, the K-Hopf algebra H_N is a Hopf-algebra form of $K[N]$, that is,

$$E \otimes_K H_N \cong E \otimes_K K[N] \cong E[N],$$

as E-Hopf algebras.

Since E/K is Galois with group G, we may use Galois descent to describe $H_N \in \mathcal{F}orm(K[N])$.

Let $F_N = \mathrm{Aut}(N)$. The E-form H_N of $K[N]$ corresponds to a 1-cocycle (homomorphism)

$$\varrho_N : G \to F_N$$

in

$$\mathrm{H}^1(G, \mathbf{Aut}(K[N])(E)) = \mathrm{H}^1(G, F_N).$$

The homomorphism ϱ_N is given as conjugation by elements of $\lambda(G)$. The kernel of ϱ_N is a normal subgroup of G defined as

$$G_0 = \{g \in G \mid \lambda(g)\eta\lambda(g^{-1}) = \eta, \forall \eta \in N\}.$$

The quotient group G/G_0 is isomorphic to a subgroup U of F_N.

Let $E_0 = E^{G_0}$. Then E_0 is Galois extension of K with group U. So the Hopf algebra form H_N of $K[N]$ determines the data U, E_0.

And using the Pareigis induction map

$$\mathcal{T}_U : \mathcal{G}al(K, U) \to \mathcal{G}al(K, F_N),$$

we compute $\mathcal{T}_U(E_0)$, which is an F_N-Galois extension of K.

Here is a main result of [KU25]:

Theorem 6.
Let E/K be a Galois extension with group G and let $(H_N, \cdot_N)$ be a Hopf-Galois structure on E/K of type N. Then $\mathcal{T}_U(E_0)$ is the preimage of H_N under Θ

Proof. We know that $\Theta(B) = H_N$ for some B in $\mathcal{G}al(K, F_N)$. By Theorem 4, B is induced from the data $V \leq F$, L/K, i.e., $B = \mathcal{T}_V(L)$ for some V-Galois extension L/K.

The proof amounts to showing that $U = V$ and $E_0 = L$, so that

$$B = \mathcal{T}_V(L) = \mathcal{T}_U(E_0).$$

□

Now, let $(H_{N'}, \cdot_{N'})$ be some other Hopf-Galois structure on E/K corresponding to a regular subgroup $N' \leq \mathrm{Perm}(G)$ normalized by $\lambda(G)$. Then $(H_{N'}, \cdot_{N'})$ is of the *same type* as $(H_N, \cdot_N)$ if $N' \cong N$.

Researchers have studied the problem of determining the K-Hopf algebra isomorphism classes of the Hopf algebras arising from the Hopf-Galois structures on E/K of the same type.

From the work of Koch, Kohl, Truman and U., we have the following criterion.

Theorem 7 (KKTU19, Theorem 2.2).

Let E/K be a Galois extension with group G. Let $(H_N, \cdot_N)$, $(H_{N'}, \cdot_{N'})$ be Hopf-Galois structures on E/K of the same type N. Then $H_N \cong H_{N'}$ as K-Hopf algebras if and only if there exists a $\lambda(G)$-invariant isomorphism $\xi : N' \to N$.

We give an extension (of sorts) of Theorem 7.

To this end, first note that since the Hopf-Galois structures $(H_N, \cdot_N)$, $(H_{N'}, \cdot_{N'})$ are of the same type N, there exists an isomorphism of groups

$$\psi : N' \to N.$$

Set $F_N = \mathrm{Aut}(N)$, $F_{N'} = \mathrm{Aut}(N')$. Then ψ yields the isomorphism

$$\hat{\psi} : F_{N'} \to F_N.$$

H_N is a Hopf form of $K[N]$ and so corresponds to a homomorphism $\varrho_N : G \to F_N$ in $\mathrm{H}^1(G, F_N)$.

$H_{N'}$ is a Hopf form of $K[N']$ and so corresponds to a homomorphism $\varrho_{N'} : G \to F_{N'}$ in $\mathrm{H}^1(G, F_{N'})$.

But, $H_{N'}$ is also a Hopf form of $K[N]$ and corresponds to a homomorphism $\hat{\psi}\varrho_{N'} : G \to F_N$ in $\mathrm{H}^1(G, F_N)$.

Put $G_0(N) = \ker(\varrho_N)$ and $G_0(N') = \ker(\hat{\psi}\varrho_{N'})$ and $U_N = G/G_0(N) \leq F_N$ and $U_{N'} = G/G_0(N') \leq F_{N'}$.

Let $E_0(N)$ denote the fixed field of $G_0(N)$, so that U_N is its Galois group as a subgroup of F_N.

And let $E_0(N')$ be the fixed field of $G_0(N')$ so that $U_{N'}$ is its Galois group as a subgroup of $F_{N'}$.

But as a subgroup of F_N, the Galois group of $E_0(N')$ is $\hat{\psi}(U_{N'})$.

Set

$$A_{U_N} = \mathcal{T}_{U_N}(E_0(N))$$

and

$$A_{\hat{\psi}(U_{N'})} = \mathcal{T}_{\hat{\psi}(U_{N'})}(E_0(N')).$$

Then A_{U_N} is the preimage of H_N and $A_{\hat{\psi}(U_{N'})}$ is the preimage of $H_{N'}$ under the Haggenmüller-Pareigis bijection

$$\Theta : \mathcal{G}al(K, F_N) \to \mathcal{F}orm(K[N]).$$

We now have an isomorphism criterion that extends Theorem 7.

Theorem 8.
Let E/K be a Galois extension with group G. Let $(H_N, \cdot_N)$, $(H_{N'}, \cdot_{N'})$ be Hopf-Galois structures on E/K corresponding to regular subgroups N, N' of $\mathrm{Perm}(G)$, respectively, of the same type N. Let $\psi : N' \to N$ be an isomorphism. The following are equivalent:

1. $A_{U_N} \cong A_{\hat{\psi}(U_{N'})}$ *as F_N-Galois extensions of K.*
2. $H_N \cong H_{N'}$ *as K-Hopf algebras.*
3. *The 1-cocycle $\varrho_N : G \to F_N$ is cohomologous to the 1-cocycle $\hat{\psi}\varrho_{N'} : G \to F_N$.*
4. *There exists a $\lambda(G)$-invariant isomorphism $\xi : N' \to N$.*

Proof. We prove 3 $\Leftrightarrow$ 4: Suppose that $\xi : N' \to N$ is a $\lambda(G)$-invariant isomorphism. Then for all $g \in G$, $\eta' \in N'$,

$$^{g}(\xi(\eta')) = \xi(^{g}\eta'),$$

which is equivalent to

$$\varrho_N(g)(\xi(\eta')) = \xi(\varrho_{N'}(g)(\eta')).$$

Note that $\xi = \nu\psi$ for some automorphism $\nu : N \to N$ (just set $\nu = \xi\psi^{-1}$).

Let $\eta' = \xi^{-1}(\eta)$ for some $\eta \in N$. Then we obtain

$$\begin{array}{rcl} \varrho_N(g)(\eta) & = & \xi(\varrho_{N'}(g)(\xi^{-1}(\eta))). \\ & = & ((\nu\psi)\varrho_{N'}(g)(\psi^{-1}\nu^{-1}))(\eta) \\ & = & (\nu(\psi\varrho_{N'}(g)\psi^{-1})\nu^{-1})(\eta) \\ & = & (\nu(\hat{\psi}\varrho_{N'})(g)\nu^{-1})(\eta), \end{array}$$

for all $g \in G$, and so ϱ_N is cohomologous to $\hat{\psi}\varrho_{N'}$.

Conversely, suppose that ϱ_N is cohomologous to $\hat{\psi}\varrho_{N'}$, i.e., suppose that there exists a fixed $\nu \in F_N$ for which

$$\hat{\psi}\varrho_{N'}(g) = \nu\varrho_N(g)\nu^{-1}$$

for all $g \in G$.

Then

$$\psi\varrho_{N'}(g)\psi^{-1} = \nu\varrho_N(g)\nu^{-1},$$

and so,

$$(\nu^{-1}\psi)\varrho_{N'}(g) = \varrho_N(g)(\nu^{-1}\psi), \tag{3}$$

where $\nu^{-1}\psi : N' \to N$ is an isomorphism.

Now, from (3),

$$(\nu^{-1}\psi)(\varrho_{N'}(g)(\eta')) = \varrho_N(g)((\nu^{-1}\psi)(\eta')).$$

for $g \in G$, $\eta' \in N'$, and so,

$$(\nu^{-1}\psi)({}^g\eta') = {}^g((\nu^{-1}\psi)(\eta')).$$

Thus $\nu^{-1}\psi : N' \to N$ is a $\lambda(G)$-invariant isomorphism.

$\square$

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