Classification of the types for which every Hopf–Galois correspondence is bijective

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Hopf Algebras and Galois Module Theory

Online

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Many thanks to Griff for organizing this conference every year!

Our project actually started last year at this conference from Lorenzo's talk.

Background

Theorem (Chase-Sweedler 1969)

Let L/K be any finite extension.

For any Hopf-Galois structure H on L/K, we have an injective correspondence

$$\Phi_H : \{K\text{-Hopf subalgebras of } H\} \longrightarrow \{\text{intermediate fields of } L/K\}$$

$$H' \longmapsto L^{H'}$$

induced by the fixed field operator

$$L^{H'} = \{x \in L \mid h' \cdot x = \epsilon(h')x \text{ for all } h' \in H'\}.$$

We shall refer to the above map Φ_H as the Hopf–Galois correspondence for H.

- In this talk, we will consider the case when L/K is a Galois extension.
- **Notation.** For any group G, we shall write $\mathrm{Sym}(G)$ for its symmetric group, and

$$\begin{cases} \lambda : G \longrightarrow \mathrm{Sym}(G); & \lambda(\sigma) = (x \mapsto \sigma x) \\ \rho : G \longrightarrow \mathrm{Sym}(G); & \rho(\sigma) = (x \mapsto x\sigma^{-1}) \end{cases}$$

for its left and right regular representations, respectively.

Theorem (Greither-Pareigis 1987)

Let L/K be a finite Galois extension with Galois group G.

There is a one-to-one correspondence between

$$\left\{ \begin{array}{c} \text{regular subgroups of } \mathrm{Sym}(G) \\ \text{that are normalized by } \lambda(G) \end{array} \right\} \longrightarrow \{\text{Hopf–Galois structures on } L/K\}$$

$$N \longmapsto H_N$$

that is explicitly defined by

$$H_N = (L[N])^G = \left\{ \sum_{\eta \in N} \ell_\eta \eta \in L[N] \;\middle|\; \sigma(\ell_\eta) = \ell_{\lambda(\sigma)\eta\lambda(\sigma)^{-1}} \text{ for all } \sigma \in G \right\}.$$

Note: G acts on N via conjugation by $\lambda(G)$. The action of H_N on L is given by

$$\left(\sum_{\eta \in N} \ell_\eta \eta \right) \cdot x = \sum_{\eta \in N} \ell_\eta \eta^{-1}(1_G)(x) \text{ for all } x \in L.$$

The type of H_N is defined to be the isomorphism class of the regular subgroup N.

The classical and canonical non-classical structures

- Let L/K be a finite Galois extension with Galois group G.

$$\left\{\begin{array}{c}\text{regular subgroups of } \mathrm{Sym}(G) \\ \text{that are normalized by } \lambda(G)\end{array}\right\} \leftrightsquigarrow \{\text{Hopf–Galois structures on } L/K\}$$

- Obviously, the images of the left and right regular representations

$$\begin{cases} \lambda : G \longrightarrow \mathrm{Sym}(G); & \lambda(\sigma) = (x \mapsto \sigma x) \\ \rho : G \longrightarrow \mathrm{Sym}(G); & \rho(\sigma) = (x \mapsto x\sigma^{-1}) \end{cases}$$

of G are regular subgroups of $\mathrm{Sym}(G)$ that are normalized by $\lambda(G)$.

Definition

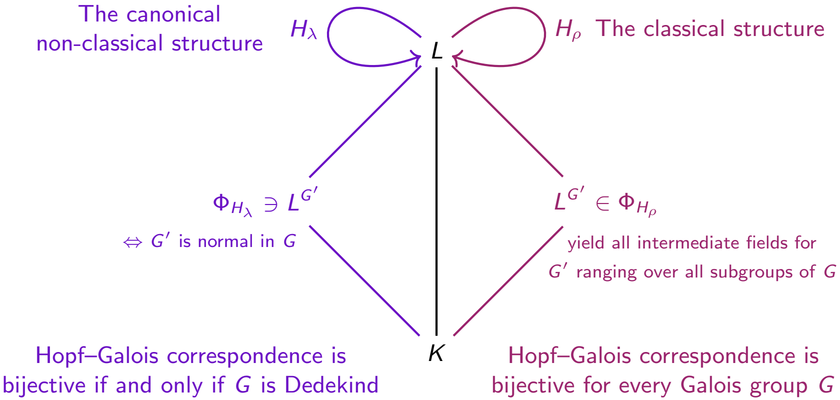
The classical structure on L/K is the Hopf–Galois structure $H_\rho := H_{\rho(G)}$.

The canonical non-classical structure on L/K is the Hopf–Galois structure $H_\lambda := H_{\lambda(G)}$.

- **Remark 1.** In the case that G is abelian, of course $\rho(G) = \lambda(G)$, so the classical and canonical non-classical structures coincide.

- **Remark 2.** The classical structure can be identified with the group ring $K[G]$, and the Hopf–Galois correspondence for H_ρ is the usual Galois correspondence.

- Let L/K be a finite Galois extension with Galois group G.



Theorem (Dedekind) The original slides contained a mistake here. I am very grateful to Andrea Caranti for pointing it out.

A finite non-abelian group G is Dedekind if and only if $G \simeq Q_8 \times T \times O$, where Q_8 is the quaternion group, T is an elementary 2-group, and O is an abelian group of odd order.

Bijectivity of the Hopf–Galois correspondence

Problem I: Fix the Galois group G

- It is natural to ask whether the Hopf–Galois correspondence, which is only injective in general, is actually bijective for a given Hopf–Galois structure.
- In Lorenzo's talk

  Classifying Galois extensions with Childs's property

  last year at this conference, he fixed the Galois group G and posed the following:

Problem I

Classify the finite groups G such that for any Galois G-extension L/K, the Hopf–Galois correspondence is bijective for any Hopf–Galois structure on L/K.

- **Note.** By the canonical non-classical structure, clearly G must be Dedekind.

Theorem I (Stefanello & Trappeniers 2023)

Fix a finite group G that is going to be the Galois group. The following are equivalent:

1. For any Galois G-extension L/K, the Hopf–Galois correspondence is bijective for any Hopf–Galois structure on L/K.
2. The group G is cyclic and $q \nmid p - 1$ for all prime divisors p, q of $|G|$.

Problem II: Fix the type N of the Hopf–Galois structure

- After Lorenzo's talk, I asked him:

  What if one fixes the type of the Hopf–Galois structure instead?

  In other words, is it possible to solve the following:

Problem II

Classify the finite groups N such that for any Hopf–Galois structure H of type N on any Galois extension L/K, the Hopf–Galois correspondence is bijective for H.

- **Note.** By the canonical non-classical structure, clearly N must be Dedekind.

Theorem II (Stefanello & T. 2025)

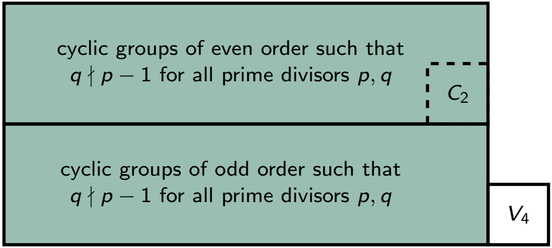
Fix a finite group N that is going to be the type. The following are equivalent:

1. For any Hopf–Galois structure H of type N on any Galois extension, the Hopf–Galois correspondence for H is bijective.
2. The group N is either isomorphic to C_2 or V_4, or is cyclic of odd order and $q \nmid p-1$ for all prime divisors p, q of $|N|$.

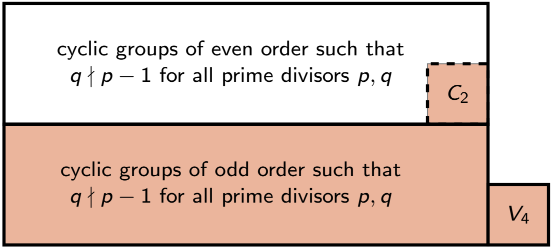
- The conditions on N are similar to but not quite the same as those of G.

Comparison of the two results

The finite groups G such that for any Galois G-extension L/K, the Hopf–Galois correspondence is bijective for any Hopf–Galois structure on L/K



The finite groups N such that for any Hopf–Galois structure H of type N on any Galois extension L/K, the Hopf–Galois correspondence for H is bijective



The key technique used in the proof

Hopf–Galois structures and skew braces

- Both of the theorems make use of the connection between Hopf–Galois structures and skew braces in the proof.

Definition

A skew brace is a set $A = (A, +, \circ)$ equipped with two binary operations such that

1. $(A, +)$ is a group. (the additive group)
2. $(A, \circ)$ is a group. (the circle/adjoint/multiplicative group)
3. The brace relation $a \circ (b + c) = (a \circ b) - a + (a \circ c)$ holds for all $a, b, c \in A$.

- Given any group $(A, \circ)$, there are two obvious ways to define the $+$ operation such that $(A, +, \circ)$ is a skew brace.
  1. The same operation: $a + b = a \circ b$ for all $a, b \in A$.
  2. The opposite operation: $a + b = a \circ^{\mathrm{op}} b = b \circ a$ for all $a, b \in A$.

Definition

A skew brace of the form $(A, \circ, \circ)$ is said to be trivial.

A skew brace of the form $(A, \circ^{\mathrm{op}}, \circ)$ is said to be almost trivial.

Hopf–Galois structures and skew braces

For any finite groups G and N of the same order:

Hopf–Galois structures of type N on a G-Galois extension

Greither-Pareigis 1987

one-to-one

regular subgroups of $\mathrm{Sym}(G)$ that are isomorphic to N and normalized by $\lambda(G)$

Galois group $\leftrightsquigarrow$ circle group

type $\leftrightsquigarrow$ additive group

Byott 1996

the operations $\circ$ on $N = (N, +)$ for which $(N, +, \circ)$ is a skew brace and $(N, \circ) \simeq G$

one-to-one

Guarnieri-Vendramin 2017

regular subgroups of $N \rtimes \mathrm{Aut}(N)$ that are isomorphic to G

- The connection between Hopf–Galois structures and skew braces was known.
- It was slightly modified and improved by Stefanello & Trappeniers.

Theorem (Stefanello-Trappeniers 2023)

Let L/K be a finite Galois extension with Galois group $G = (G, \circ)$.

There is a one-to-one correspondence between

$$\left\{ \begin{array}{c} \text{operations } + \text{ on } G \text{ such that} \\ (G, +, \circ) \text{ is a skew brace} \end{array} \right\} \longrightarrow \{\text{Hopf–Galois structures on } L/K\}$$

$$+ \longmapsto H_+$$

that is explicitly defined by

$$H_+ = (L[(G,+)])^{(G,\circ)} = \left\{ \sum_{\tau \in G} \ell_\tau \tau \in L[(G,+)] \;\middle|\; \sigma(\ell_\tau) = \ell_{\gamma_\sigma(\tau)} \text{ for all } \sigma, \tau \in G \right\}.$$

Note: $(G, \circ)$ acts on $(G, +)$ via the gamma map γ of $(G, +, \circ)$. The action of H_+ on L is given by

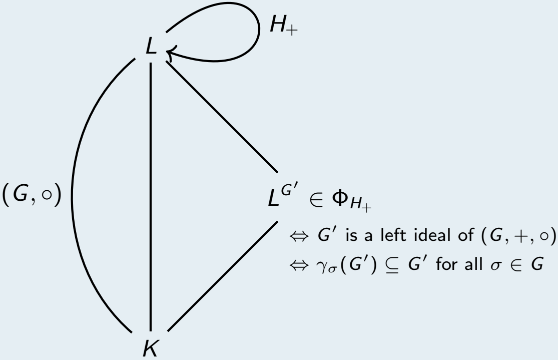
$$\left(\sum_{\tau \in G} \ell_\tau \tau \right) \cdot x = \sum_{\tau \in G} \ell_\tau \tau(x) \text{ for all } x \in L.$$

In this case the type of H_+ is the isomorphism class of the corresponding group $(G, +)$.

Theorem (Stefanello & Trappeniers 2023)

Let L/K be a finite Galois extension with Galois group $G = (G, \circ)$.

Let $+$ be an operation on G such that $(G, +, \circ)$ is a skew brace and let H_+ denote the corresponding Hopf–Galois structure on L/K. Then:

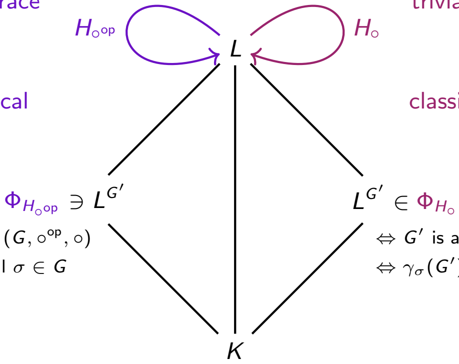


Thus, the Hopf–Galois correspondence for H_+ is bijective if and only if every subgroup G' of $(G, \circ)$ is a left ideal of $(G, +, \circ)$.

- Let L/K be a finite Galois extension with Galois group $G = (G, \circ)$.

almost trivial skew brace
$(G, \circ^{\mathrm{op}}, \circ)$
$\updownarrow$
canonical non-classical structure

trivial skew brace
$(G, \circ, \circ)$
$\updownarrow$
classical structure



$\Leftrightarrow$ G' is a left ideal of $(G, \circ^{\mathrm{op}}, \circ)$
$\Leftrightarrow$ $\gamma_\sigma(G') \subseteq G'$ for all $\sigma \in G$

$\Leftrightarrow$ G' is a left ideal of $(G, \circ, \circ)$
$\Leftrightarrow$ $\gamma_\sigma(G') \subseteq G'$ for all $\sigma \in G$

- For the trivial skew brace, we have $\gamma_\sigma(\tau) = \sigma^{-1} \circ (\sigma \circ \tau) = \tau$.
  $\therefore L^{G'} \in \Phi_{H_\circ}$ for all subgroups G' of G. $\Phi_{H_\circ}$ is always bijective.
- For the almost trivial skew brace, we have $\gamma_\sigma(\tau) = \sigma^{-1} \circ^{\mathrm{op}} (\sigma \circ \tau) = \sigma \circ \tau \circ \sigma^{-1}$.
  $\therefore L^{G'} \in \Phi_{H_{\circ^{\mathrm{op}}}}$ if and only if G' is normal in G. $\Phi_{H_{\circ^{\mathrm{op}}}}$ is bijective $\Leftrightarrow$ G is Dedekind.

Skew-brace-theoretic versions of the proposed problems

Problem I

Classify the finite groups G such that for any Galois G-extension L/K, the Hopf–Galois correspondence is bijective for any Hopf–Galois structure on L/K.

Problem I: skew-brace-theoretic version

Classify the finite groups $G = (G, \circ)$ such that for any group operation $+$ on G making $(G, +, \circ)$ into a skew brace, every subgroup of $(G, \circ)$ is a left ideal of $(G, +, \circ)$.

- Theorem I (Stefanello & Trappeniers 2023) was proven using this reduction.

Problem II

Classify the finite groups N such that for any Hopf–Galois structure H of type N on any Galois extension L/K, the Hopf–Galois correspondence is bijective for H.

Problem II: skew-brace-theoretic version

Classify the finite groups $N = (N, +)$ such that for any group operation $\circ$ on N making $(N, +, \circ)$ into a skew brace, every subgroup of $(N, \circ)$ is a left ideal of $(N, +, \circ)$.

- Theorem II (Stefanello & T. 2025) was proved using the reduction.

Outline of the proof of Theorem II

A reduction lemma

Theorem II (Stefanello & T. 2025)

Fix a finite group N that is going to be the type. The following are equivalent:

- (i) For any Hopf–Galois structure H of type N on any Galois extension, the Hopf–Galois correspondence for H is bijective.
- (ii) The group N is either isomorphic to C_2 or V_4, or is cyclic of odd order and $q \nmid p-1$ for all prime divisors p, q of $|N|$.

Theorem II: skew-brace-theoretic version (Stefanello & T. 2025)

Fix a finite group $N = (N, +)$ that is going to be the type. The following are equivalent:

- (i) For any operation $\circ$ on N making $(N, +, \circ)$ into a skew brace, every subgroup of $(N, \circ)$ is a left ideal of $(N, +, \circ)$.
- (ii) The group N is either isomorphic to C_2 or V_4, or is cyclic of odd order and $q \nmid p-1$ for all prime divisors p, q of $|N|$.

Lemma

If M does not satisfy (i), then $M \times M'$ does not satisfy (i) for all groups M'.

The implication (i) $\Rightarrow$ (ii)

Theorem II: skew-brace-theoretic version (Stefanello & T. 2025)

Fix a finite group $N = (N, +)$ that is going to be the type. The following are equivalent:

1. For any operation $\circ$ on N making $(N, +, \circ)$ into a skew brace, every subgroup of $(N, \circ)$ is a left ideal of $(N, +, \circ)$.
2. The group N is either isomorphic to C_2 or V_4, or is cyclic of odd order and $q \nmid p - 1$ for all prime divisors p, q of $|N|$.

- Recall that either N is abelian or N is a direct product of Q_8 with an abelian group.
- One can show that the following groups do not satisfy (i).
  - a. Q_8 $\rightsquigarrow$ abelian
  - b. $C_2 \times C_2 \times C_2$ $\rightsquigarrow$ C_2 or V_4 or not elementary 2-abelian
  - c. C_{2n} for any $n \geq 2$ $\rightsquigarrow$ C_2 or V_4 or odd order
  - d. $C_{p^n} \times C_{p^m}$ for any odd primes p and m, n $\rightsquigarrow$ C_2 or V_4 or cyclic of odd order
  - e. $C_{p^n} \times C_{q^m}$ for any primes p, q with $q \mid p - 1$ and m, n
- It follows that N must satisfy (ii).

The implication ⓘⓘ $\Rightarrow$ ⓘ

Theorem II: skew-brace-theoretic version (Stefanello & T. 2025)

Fix a finite group $N = (N, +)$ that is going to be the type. The following are equivalent:

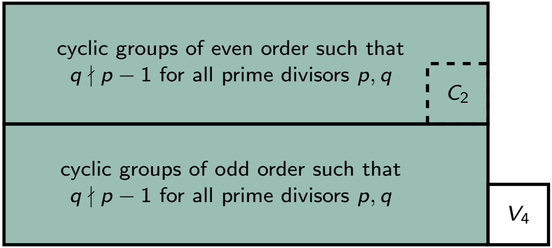
1. For any operation $\circ$ on N making $(N, +, \circ)$ into a skew brace, every subgroup of $(N, \circ)$ is a left ideal of $(N, +, \circ)$.
2. The group N is either isomorphic to C_2 or V_4, or is cyclic of odd order and $q \nmid p - 1$ for all prime divisors p, q of $|N|$.

- The case $N \simeq C_2$ is obvious.
- The case $N \simeq V_4$ can also be dealt with very easily.
- The case $N \simeq C_n$ with $q \nmid p - 1$ for all prime divisors p, q of n:
  - a. $(N, \circ)$ is a C-group, i.e. the Sylow subgroups are all cyclic. (Rump 2019)
  - b. $(N, \circ) \simeq C_d \rtimes C_e$ for some $\gcd(d, e) = 1$ (Murty & Murty 1984)
  - c. $(N, \circ) \simeq C_n \simeq (N, +)$ (the hypothesis on the prime divisors of n)
- It then follows that N must satisfy ⓘ. See Remark 2.9 of the paper for the details.

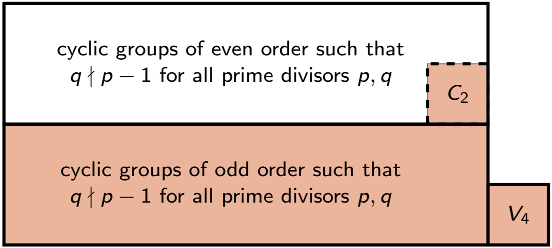
The difference in the Klein four-group

Comparison of the two results

The finite groups G such that for any Galois G-extension L/K, the Hopf–Galois correspondence is bijective for any Hopf–Galois structure on L/K



The finite groups N such that for any Hopf–Galois structure H of type N on any Galois extension L/K, the Hopf–Galois correspondence for H is bijective



The two non-trivial skew braces of order four

- First non-trivial skew brace: $(A,+) \simeq V_4$ and $(A,\circ) \simeq C_4$

$$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, +, \circ): \quad \vec{x} \circ \vec{y} = \left[\begin{smallmatrix} x_1+y_1+x_2y_2 \\ x_2+y_2 \end{smallmatrix}\right], \ \ \gamma_{\vec{x}}(\vec{y}) = \left[\begin{smallmatrix} 1 & x_2 \\ 0 & 1 \end{smallmatrix}\right] \vec{y}$$

There is only one non-trivial proper subgroup of $(A,\circ)$:

$$\{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\}$$

This is a left ideal of $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, +, \circ)$.

- Second non-trivial skew brace: $(A,+) \simeq C_4$ and $(A,\circ) \simeq V_4$

$$(\mathbb{Z}/4\mathbb{Z}, +, \circ): \quad x \circ y = x + y + 2xy, \ \ \gamma_x(y) = (1+2x)y$$

There are three non-trivial proper subgroups of $(A,\circ)$:

$$\{0,1\}, \quad \{0,2\}, \quad \{0,3\}$$

Only $\{0,2\}$ is a left ideal of $(\mathbb{Z}/4\mathbb{Z}, +, \circ)$.

$\boxed{\text{Thus, we have to exclude } V_4 \text{ when we are fixing the Galois group.}}$

References in order of appearance

1. S. U. Chase and M. E. Sweedler, *Hopf Algebras and Galois Theory*, Lecture Notes in Mathematics, Vol. 97. Springer-Verlag, Berlin-New York, 1969.
2. C. Greither and B. Pareigis, *Hopf Galois theory for separable field extensions*, J. Algebra 106 (1987), no. 1, 239–258.
3. L. Stefanello and S. Trappeniers, *On the connection between Hopf–Galois structures and skew braces*, Bull. Lond. Math. Soc. 55 (2023), no. 4, 1726–1748.
4. N. P. Byott, *Galois structure for separable field extension*, Comm. Algebra 24 (1996), no. 10, 3217–3228. Corrigendum, *ibid.* no. 11, 3705.
5. L. Guarnieri and L. Vendramin, *Skew braces and the Yang–Baxter equation*, Math. Comp. 86 (2017), no. 307, 2519–2534.
6. W. Rump, *Classification of cyclic braces, II*, Trans. Amer. Math. Soc. 372 (2019), no. 1, 305–328.
7. M. R. Murty and V. K. Murty, *On groups of squarefree order*, Math. Ann. 267 (1984), no. 3, 299–309.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