Some nonabelian subgroups of the Nottingham group over $\mathbb{F}_4$

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The Nottingham group and its subgroups

Let k be a finite field of characteristic p. The Nottingham group for k, denoted $\mathcal{N}(k)$, consists of all power series $\phi(t) \in k[[t]]$ of the form $\phi(t) = t + a_1 t^2 + a_2 t^3 + \cdots$, with the operation of substitution.

$\mathcal{N}(k)$ is a pro-p group. It follows from a theorem of Witt [Wi36] that every finite p-group G is isomorphic to a subgroup of $\mathcal{N}(k)$.

Problem: For a finite p-group G, find representatives for the conjugacy classes of subgroups of $\mathcal{N}(k)$ which are isomorphic to G.

Klopsch solved this problem in the case where $G \cong C_p$ is a cyclic group of order p.

Klopsch's classification

Let $a \in k^\times$ and let $m \geq 1$ be such that $p \nmid m$. Define $\psi_{m,a} \in \mathcal{N}(k)$ by

$$\begin{aligned}\psi_{m,a}(t) &= t(1 - mat^m)^{-1/m} \\ &= t + at^{m+1} + \cdots .\end{aligned}$$

For $r \in \mathbb{Z}$ we have $\psi_{m,a}^{\circ r} = t(1 - rmat^m)^{-1/m}$. Hence $\psi_{m,a}$ has order p.

Theorem (Klopsch [Kl00])

For each $\phi \in \mathcal{N}(k)$ with order p there are uniquely determined $a \in k^\times$ and $m \geq 1$ with $p \nmid m$ such that ϕ is conjugate to $\psi_{m,a}$.

This leads to an explicit classification of the conjugacy classes of cyclic subgroups of $\mathcal{N}(k)$ of order p.

Finding formulas for elements of more complicated finite subgroups of $\mathcal{N}(k)$ has proven to be difficult.

For instance, the only cases for which elements of $\mathcal{N}(k)$ of order p^d with $d \geq 2$ have been explicitly constructed are those with $p = 2$ and $d = 2$.

Finite automata and elements of $\mathcal{N}(k)$

It was proposed by Byszewski, Cornelissen, and Tijsma [BCT22] that elements of finite subgroups of $\mathcal{N}(k)$ should be described in terms of finite automata.

The basis for this approach is Christol's theorem, which says that $\sum_{n=0}^{\infty} a_n t^n \in k[[t]]$ is algebraic over the rational function field $k(t)$ if and only if the coefficient sequence (a_n) is the output of a finite automaton of an appropriate type.

In [BCT22], automata are used to construct abelian subgroups of Nottingham groups. Specifically, subgroups of $\mathcal{N}(\mathbb{F}_2)$ which are isomorphic to C_4, C_8, and $C_2 \times C_2$ are constructed there.

In this talk I will show how to use automata to construct subgroups of $\mathcal{N}(\mathbb{F}_4)$ which are isomorphic to the nonabelian groups Q_8 and D_4.

Finite automata

We wish to describe elements of $\mathcal{N}(k)$ using finite automata.

Suppose $k \cong \mathbb{F}_q$. Then our automata will use $\{0, 1, 2, \ldots, q-1\}$ as input alphabet and k as output alphabet.

An automaton is a directed multigraph with loops such that every vertex has outdegree q. The edges and vertices of the graph have labels with the following properties:

- The edges emanating from a vertex v are labeled with the elements of the input alphabet $\{0, 1, 2, \ldots, q-1\}$, with each label appearing exactly once.
- The vertices are labeled with elements of the output alphabet k.
- If an edge with label 0 connects v_1 to v_2 then the vertices v_1, v_2 must have the same label.

In addition, one of the vertices should be marked "Start".

The vertices of the automaton are also referred to as "states".

Automatic sequences

An automaton can be used to generate an "automatic sequence" as follows:

Let $n \geq 0$ and write n in base q as

$$n = d_r d_{r-1} \dots d_1 d_0 = d_0 q^0 + d_1 q^1 + \cdots + d_r q^r.$$

Beginning at the Start vertex, trace the path though the digraph given by the edges labeled $d_0, d_1, \dots, d_{r-1}, d_r$.

The nth term in our automatic sequence is the label attached to the final vertex in this path. Thanks to the restriction on the vertex labeling, adding leading 0s to the base-q representation of n doesn't change the output.

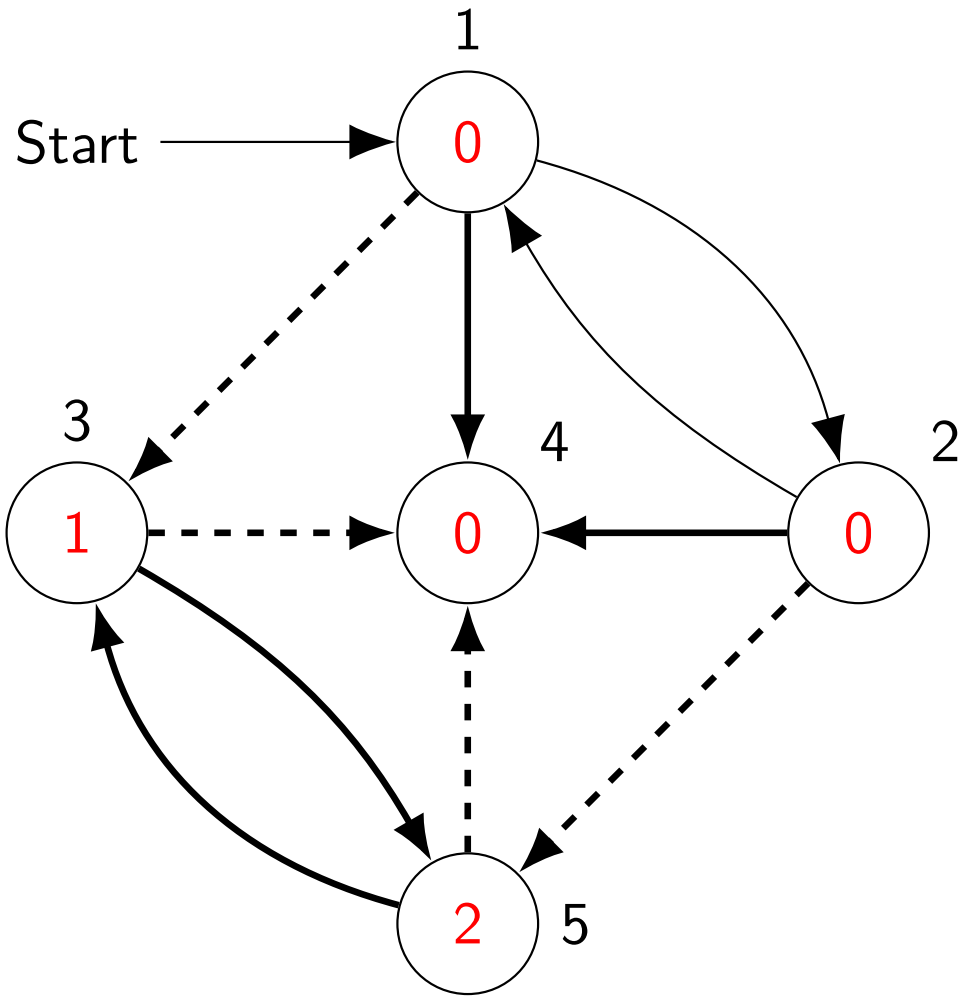
Theorem (Christol [Ch79])

$\sum_{n=0}^{\infty} a_n t^n \in k[[t]]$ is algebraic over $k(t)$ if and only if $(a_n)_{n \geq 0}$ is an automatic sequence of the type described above.

A finite automaton

Input alphabet: $\{0, 1, 2\}$
Output alphabet: $\mathbb{F}_3 = \{0, 1, 2\}$

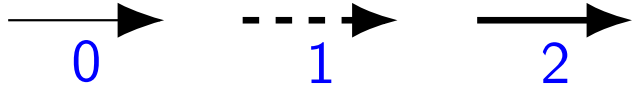
State	0	1	2	label
1	2	3	4	0
2	1	5	4	0
3	3	4	5	1
4	4	4	4	0
5	5	4	3	2



Directed edges leading from a state to itself are not drawn.

This automaton produces the sequence of coefficients of the order-3 Klopsch series $\psi_{2,2} \in \mathcal{N}(\mathbb{F}_3)$:

$$t(1-t^2)^{-1/2} = t+2t^3+2t^7+t^9+O(t^{19}).$$



The group $\mathcal{A}(k)$ and automorphisms of local fields

The set

$$\mathcal{A}(k) = \{a_0 t + a_1 t^2 + a_2 t^3 + \cdots : a_i \in k,\ a_0 \neq 0\},$$

with the operation of substitution forms a group. There is an exact sequence

$$1 \longrightarrow \mathcal{N}(k) \longrightarrow \mathcal{A}(k) \longrightarrow k^\times \longrightarrow 1.$$

Let E be a local field of characteristic p with residue field k; thus $E \cong k((x))$.

Let π_E be a uniformizer for E and let $\sigma \in \mathrm{Aut}_k(E)$. Then there is a uniquely determined $\phi_\sigma(t) \in \mathcal{A}(k)$ such that $\sigma(\pi_E) = \phi_\sigma(\pi_E)$.

The map $\theta_{\pi_E} : \mathrm{Aut}_k(E) \to \mathcal{A}(k)$ defined by $\theta_{\pi_E}(\sigma) = \phi_\sigma$ is an anti-isomorphism.

The Nottingham group and wild automorphisms

Say $\sigma \in \mathrm{Aut}_k(E)$ is a wild automorphism of E if $\sigma(\pi_E) + \mathcal{M}_E^2 = \pi_E + \mathcal{M}_E^2$.

The wild automorphisms of E form a subgroup $\mathrm{Aut}_k^1(E)$ of $\mathrm{Aut}_k(E)$. The anti-isomorphism $\theta_{\pi_E} : \mathrm{Aut}_k(E) \to \mathcal{A}(k)$ maps $\mathrm{Aut}_k^1(E)$ onto $\mathcal{N}(k)$.

Let F be a subfield of E such that E/F is a finite totally ramified Galois p-extension. Then $\mathrm{Gal}(E/F) \leq \mathrm{Aut}_k^1(E)$ and $\theta_{\pi_E}(\mathrm{Gal}(E/F))$ is a finite subgroup of $\mathcal{N}(k)$.

Conversely, given a finite subgroup $\mathcal{G}$ of $\mathcal{N}(k)$, set $G = \theta_{\pi_E}^{-1}(\mathcal{G})$. Then E/E^G is a finite totally ramified Galois p-extension.

Depth and ramification

Define the depth of $\phi \in \mathcal{N}(k)$ by $D(\phi) = v_t(\phi(t) - t) - 1$.

Let $\mathcal{G}$ be a finite subgroup of $\mathcal{N}(k)$. For $n \geq 1$ define $\mathcal{G}_n = \{\phi \in \mathcal{G} : D(\phi) \geq n\}$. Then $\mathcal{G}_n$ is a normal subgroup of $\mathcal{G}$, known as the nth ramification subgroup of $\mathcal{G}$.

We say that $b \geq 1$ is a ramification break of $\mathcal{G}$ if $\mathcal{G}_b \neq \mathcal{G}_{b+1}$.

If $|\mathcal{G}_b : \mathcal{G}_{b+1}| = p^m$ say that b is a ramification break with multiplicity m.

Let $|\mathcal{G}| = p^n$. Then $\mathcal{G}$ has n ramification breaks $b_1 \leq b_2 \leq \cdots \leq b_n$, counted with multiplicities.

Let E/F be a finite totally ramified Galois p-extension. Set $\mathcal{G} = \theta_{\pi_E}(\mathrm{Gal}(E/F))$. Then $\mathcal{G} \leq \mathcal{N}(k)$.

The ramification filtration on $\mathcal{G}$ induces a ramification filtration on $\mathrm{Gal}(E/F)$. We define the ramification breaks of $\mathrm{Gal}(E/F)$ (or of E/F) to be the same as the ramification breaks of $\mathcal{G}$.

A category of field extensions

Let k be a finite field. We define a category $\mathcal{C}_k$ whose objects are pairs $(i : F \to E, \pi_F + \mathcal{M}_F^2)$, where

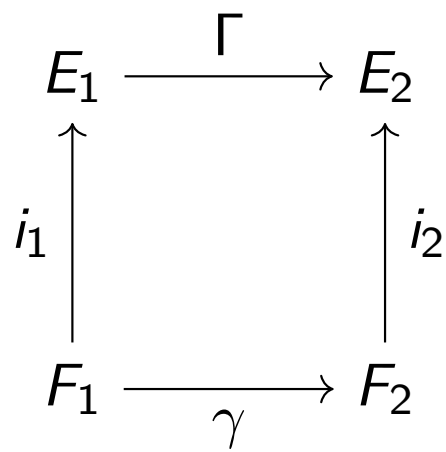
- F, E are local fields of characteristic p with residue field k,
- i is a k-algebra embedding such that $E/i(F)$ is a Galois p-extension.
- $\pi_F + \mathcal{M}_F^2$ is a generator for the $\mathcal{O}_F$-module $\mathcal{M}_F/\mathcal{M}_F^2$.

A $\mathcal{C}_k$-morphism

$$(\gamma, \Gamma) : (i_1 : F_1 \to E_1, \pi_{F_1} + \mathcal{M}_{F_1}^2) \longrightarrow (i_2 : F_2 \to E_2, \pi_{F_2} + \mathcal{M}_{F_2}^2)$$

consists of k-algebra isomorphisms $\gamma : F_1 \to F_2$ and $\Gamma : E_1 \to E_2$ such that

- $\Gamma \circ i_1 = i_2 \circ \gamma$,
- $\gamma(\pi_{F_1}) + \mathcal{M}_{F_2}^2 = \pi_{F_2} + \mathcal{M}_{F_2}^2$.



Isomorphisms of Galois groups

It follows from the definition that every $\mathcal{C}_k$-morphism is an isomorphism.

Suppose we have a $\mathcal{C}_k$-isomorphism

$$(\gamma, \Gamma) : (i_1 : F_1 \to E_1, \pi_{F_1} + \mathcal{M}_{F_1}^2) \xrightarrow{\sim} (i_2 : F_2 \to E_2, \pi_{F_2} + \mathcal{M}_{F_2}^2).$$

Then there is an isomorphism of filtered groups

$$\mathrm{Gal}(E_1/i_1(F_1)) \xrightarrow{\sim} \mathrm{Gal}(E_2/i_2(F_2))$$

which maps $\sigma \in \mathrm{Gal}(E_1/i_1(F_1))$ to $\Gamma \circ \sigma \circ \Gamma^{-1}$.

Some subcategories of $\mathcal{C}_k$

Let E be a local field of characteristic p with residue field k and let π_E be a uniformizer for E. Let $\mathcal{C}^{E,\pi_E}$ denote the full subcategory of $\mathcal{C}_k$ whose objects have the form $(i : F \to E, \pi_F + \mathcal{M}_F^2)$, where E/F is a totally ramified Galois p-extension, $i : F \to E$ is inclusion, and $\pi_F = \mathrm{N}_{E/F}(\pi_E)$.

A morphism from $(i_1 : F_1 \to E, \pi_{F_1} + \mathcal{M}_{F_1}^2)$ to $(i_2 : F_2 \to E, \pi_{F_2} + \mathcal{M}_{F_2}^2)$ is a wild automorphism of E which maps F_1 onto F_2.

Let K be a local field of characteristic p with residue field k and let π_K be a uniformizer for K. Let $\mathcal{C}_{K,\pi_K}$ denote the full subcategory of $\mathcal{C}_k$ whose objects are of the form $(i : K \to L, \pi_K + \mathcal{M}_K^2)$, where L/K is a totally ramified Galois p-extension and $i : K \to L$ is inclusion.

A morphism from $(i_1 : K \to L_1, \pi_K + \mathcal{M}_K^2)$ to $(i_2 : K \to L_2, \pi_K + \mathcal{M}_K^2)$ is an isomorphism from L_1 to L_2 which induces a wild automorphism of K.

Since all our fields are isomorphic to $k((t))$, every object in $\mathcal{C}_k$ is isomorphic to an object in $\mathcal{C}^{E,\pi_E}$, and to an object in $\mathcal{C}_{K,\pi_K}$. Hence the inclusions of $\mathcal{C}^{E,\pi_E}$ and $\mathcal{C}_{K,\pi_K}$ into $\mathcal{C}_k$ are equivalences of categories.

Subgroups of $\mathcal{N}(k)$ and field extensions

Let G and H be finite subgroups of $\mathrm{Aut}_k^1(E)$. Then $\theta_{\pi_E}(G)$ is conjugate to $\theta_{\pi_E}(H)$ in $\mathcal{N}(k)$ if and only if the objects in $\mathcal{C}^{E,\pi_E}$ which correspond to G and H are isomorphic, i.e.,

$$(i : E^G \to E, \mathrm{N}_{E/E^G}(\pi_E) + \mathcal{M}_{E^G}^2) \cong (i : E^H \to E, \mathrm{N}_{E/E^H}(\pi_E) + \mathcal{M}_{E^H}^2).$$

Therefore we get:

Proposition

Let G be a finite p-group, let K be a local field of characteristic p with residue field k, and let π_K be a uniformizer of K. Then there is a one-to-one correspondence between

- *Isomorphism classes of objects $(i : K \to L, \pi_K + \mathcal{M}_K^2) \in \mathcal{C}_{K,\pi_K}$ such that L/K is a G-extension.*
- *Conjugacy classes of subgroups of $\mathcal{N}(k)$ which are isomorphic to G.*

This bijection maps the isomorphism class of $(i : K \to L, \pi_K + \mathcal{M}_K^2)$ to the conjugacy class represented by $\theta_{\pi_L}(Gal(L/K))$, where π_L is any uniformizer of L such that $\mathrm{N}_{L/K}(\pi_L) + \mathcal{M}_K^2 = \pi_K + \mathcal{M}_K^2$.

Elementary abelian extensions of $\mathbb{F}_4((u))$

Let Q_8 be the quaternion group and let D_4 be the dihedral group of order 8. We wish to construct totally ramified Q_8-extensions and D_4-extensions of K with minimum ramification breaks.

As a first step, we construct a totally ramified $(C_2 \times C_2)$-extension with minimum ramification breaks.

Let $k = \mathbb{F}_4$ be the finite field with 4 elements and let $s \in \mathbb{F}_4 \setminus \mathbb{F}_2$. Then $s^2 + s + 1 = 0$ and $\mathbb{F}_4 = \{0, 1, s, s^2\}$. Let $K = \mathbb{F}_4((u))$ be the field of formal Laurent series over $\mathbb{F}_4$.

Proposition

There is a single $\mathcal{C}_k$-isomorphism class of pairs

$$(i : K \to M, u + \mathcal{M}_K^2) \in \mathcal{C}_{K,u},$$

such that M/K is a totally ramified $(C_2 \times C_2)$-extension with ramification breaks 1,1.

An explicit description of M

We define the Artin-Schreier operator on fields of characteristic $p = 2$ by $\wp(x) = x^p - x = x^2 - x$.

Recall that $K = \mathbb{F}_4((u))$. Let $\alpha_1, \alpha_2 \in K^{sep}$ satisfy $\wp(\alpha_1) = su^{-1}$ and $\wp(\alpha_2) = s^2u^{-1}$. Then $M = K(\alpha_1, \alpha_2)$ is a totally ramified $(C_2 \times C_2)$-extension of K with ramification breaks 1,1.

Lemma

Set $y = s\alpha_1 + s^2\alpha_2$. Then

- $\alpha_1 = \wp(sy) = s^2y^2 + sy$,
- $\alpha_2 = \wp(s^2y) = sy^2 + s^2y$.
- $u^{-1} = \wp(\wp(y)) = y^4 + y$,

It follows that $v_M(y) = -1$, so y^{-1} is a uniformizer for M.

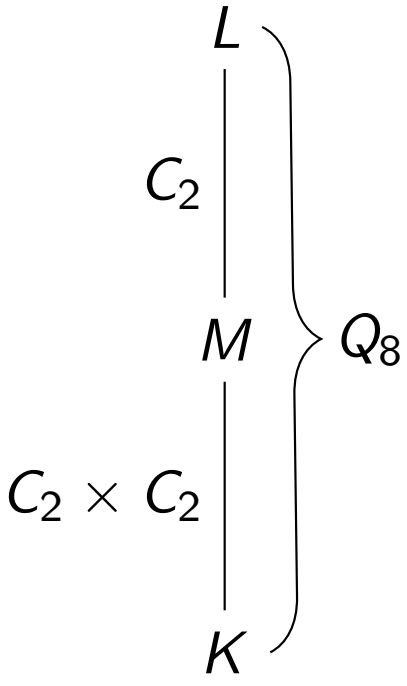
Let $\overline{\sigma}_1, \overline{\sigma}_2 \in \mathrm{Gal}(M/K)$ be defined by $\overline{\sigma}_1(\alpha_1) = \alpha_1 + 1$, $\overline{\sigma}_1(\alpha_2) = \alpha_2$, $\overline{\sigma}_2(\alpha_1) = \alpha_1$, and $\overline{\sigma}_2(\alpha_2) = \alpha_2 + 1$.

Then $\overline{\sigma}_1(y) = y + s$ and $\overline{\sigma}_2(y) = y + s^2$.

Quaternion extensions of K

Proposition

Let M/K be a totally ramified $(C_2 \times C_2)$-extension with ramification breaks 1,1 and write $M = K(y)$ as above. There are precisely two C_2-extensions L/M such that L/K is a totally ramified Q_8-extension with ramification breaks $1, 1, 3$. These are generated over M by the roots of $X^2 - X - y^3 - \delta$, with $\delta \in \{0, s\}$.



Corollary

There are two conjugacy classes of subgroups of $\mathcal{N}(\mathbb{F}_4)$ which are isomorphic to Q_8 and have ramification breaks $1, 1, 3$.

A uniformizer for L

Assume for now that $\delta = 0$.

Let $\alpha_3 \in K^{sep}$ be a root of $X^2 - X - y^3$. Then $L = M(\alpha)$ is a totally ramified Q_8-extension of K.

Since $\wp(\alpha_3) = y^3$ has M-valuation -3, we get $v_L(\alpha_3) = -3$.

Set $t = y/\alpha_3$. Then $v_L(t) = -2 - (-3) = 1$, so t is a uniformizer for L.

Recall that $\overline{\sigma}_1 \in \mathrm{Gal}(M/K)$ satisfies $\overline{\sigma}_1(y) = y + s$. We find that $\overline{\sigma}_1$ extends to $\sigma_1 \in \mathrm{Gal}(L/K)$ such that $\sigma_1(\alpha_3) = \alpha_3 + s^2 y + s^2$.

Similarly, we may extend $\overline{\sigma}_2 \in \mathrm{Gal}(M/K)$ to $\sigma_2 \in \mathrm{Gal}(L/K)$ by setting $\sigma_2(\alpha_3) = \alpha_3 + sy + s$.

Since $\alpha_3 t = y$ we get

$$(\alpha_3 + s^2 y + s^2)\sigma_1(t) = y + s$$
$$(\alpha_3 + sy + s)\sigma_2(t) = y + s^2.$$

A quaternion subgroup of $\mathcal{N}(\mathbb{F}_4)$

We've shown that $y, \alpha_3, t, \sigma_1(t)$ satisfy 3 polynomial equations over $\mathbb{F}_4$:

$$\alpha_3^2 - \alpha_3 = y^3, \quad t\alpha_3 = y, \quad (\alpha_3 + s^2 y + s^2)\sigma_1(t) = y + s.$$

We want to deduce from these a polynomial relation between t and $X = \sigma_1(t)$. This relation (if it exists) will be an element of the following ideal in the polynomial ring $\mathbb{F}_4[y, \alpha_3, t, X]$:

$$J = (\alpha_3^2 - \alpha_3 - y^3, t\alpha_3 - y, (\alpha_3 + s^2 y + s^2)X - y - s).$$

Using Magma we find a Gröbner basis for J using an elimination term order. We find that $\sigma_1(t)$ is a root of

$$f_{\sigma_1}(t, X) = (t^2 + 1)X^2 + X + st^2 + t.$$

Similarly, $\sigma_2(t)$ is a root of

$$f_{\sigma_2}(t, X) = (t^2 + 1)X^2 + X + s^2 t^2 + t.$$

We can use these to compute terms of $\sigma_1(t)$ and $\sigma_2(t)$ recursively:

$$\sigma_1(t) = t + s^2 t^2 + s^2 t^4 + st^6 + st^8 + st^{10} + s^2 t^{12} + s^2 t^{14} + s^2 t^{16} + O(t^{18})$$

$$\sigma_2(t) = t + st^2 + st^4 + s^2 t^6 + s^2 t^8 + s^2 t^{10} + st^{12} + st^{14} + st^{16} + O(t^{18}).$$

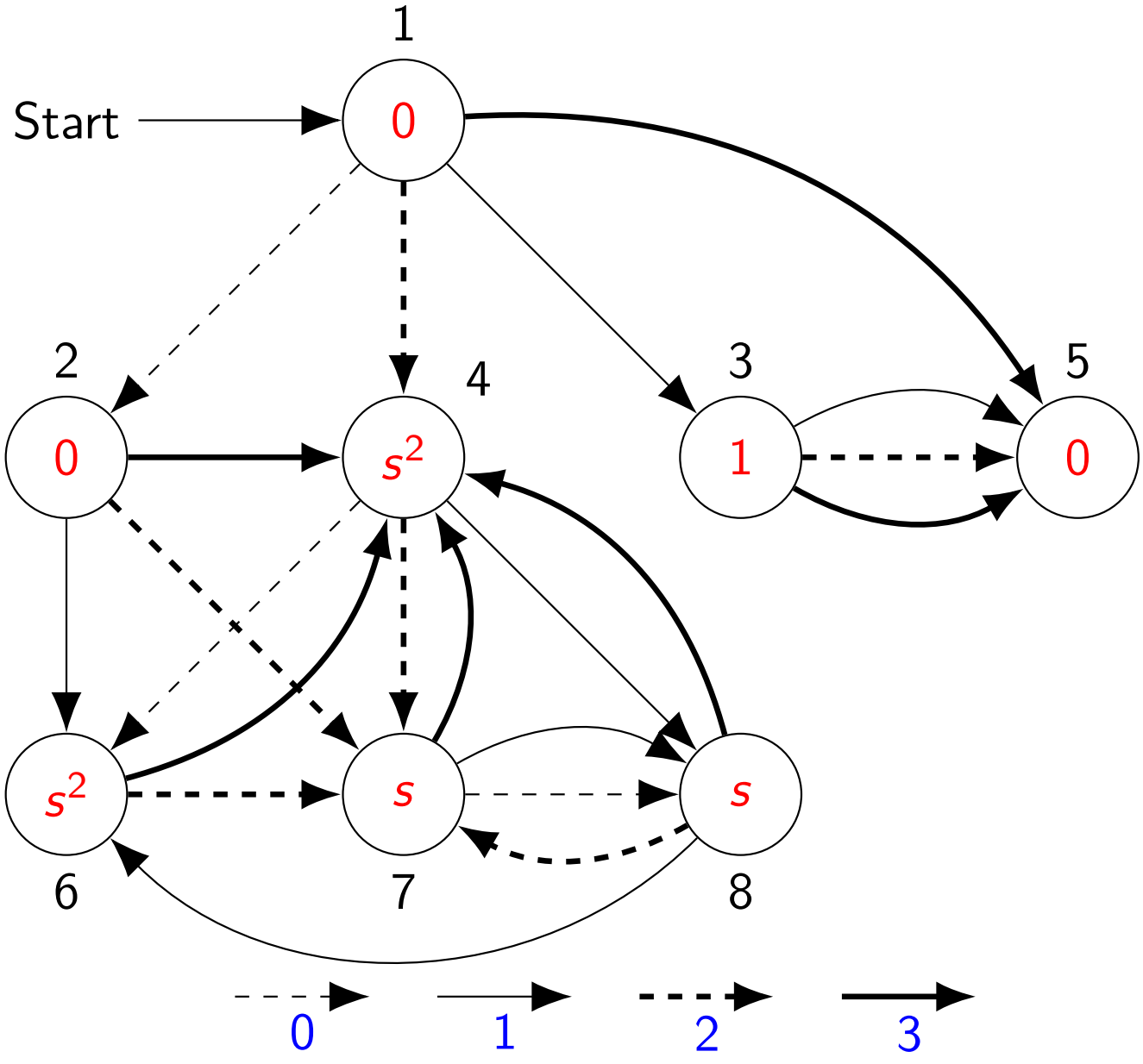
Automata for σ_1 and σ_2

It follows from the preceding slide that $\sigma_1(t)$, $\sigma_2(t)$ are algebraic over $\mathbb{F}_4(t)$. Hence by Christol's theorem, the sequence of coefficients of $\sigma_i(t)$ is the output of a finite automaton.

We can apply Algorithm 3.2.3 of [BCT22] to construct these automata. The automata for $\sigma_1(t)$ and $\sigma_2(t)$ have the same digraph and the same edge labels. The state labels are determined by the terms in the expansions for $\sigma_i(t)$ that we computed above.

State	0	1	2	3	σ_1 label	σ_2 label
1	2	3	4	5	0	0
2	2	6	7	4	0	0
3	3	5	5	5	1	1
4	6	8	7	4	s^2	s
5	5	5	5	5	0	0
6	6	6	7	4	s^2	s
7	8	8	7	4	s	s^2
8	8	6	7	4	s	s^2

Automaton for σ_1

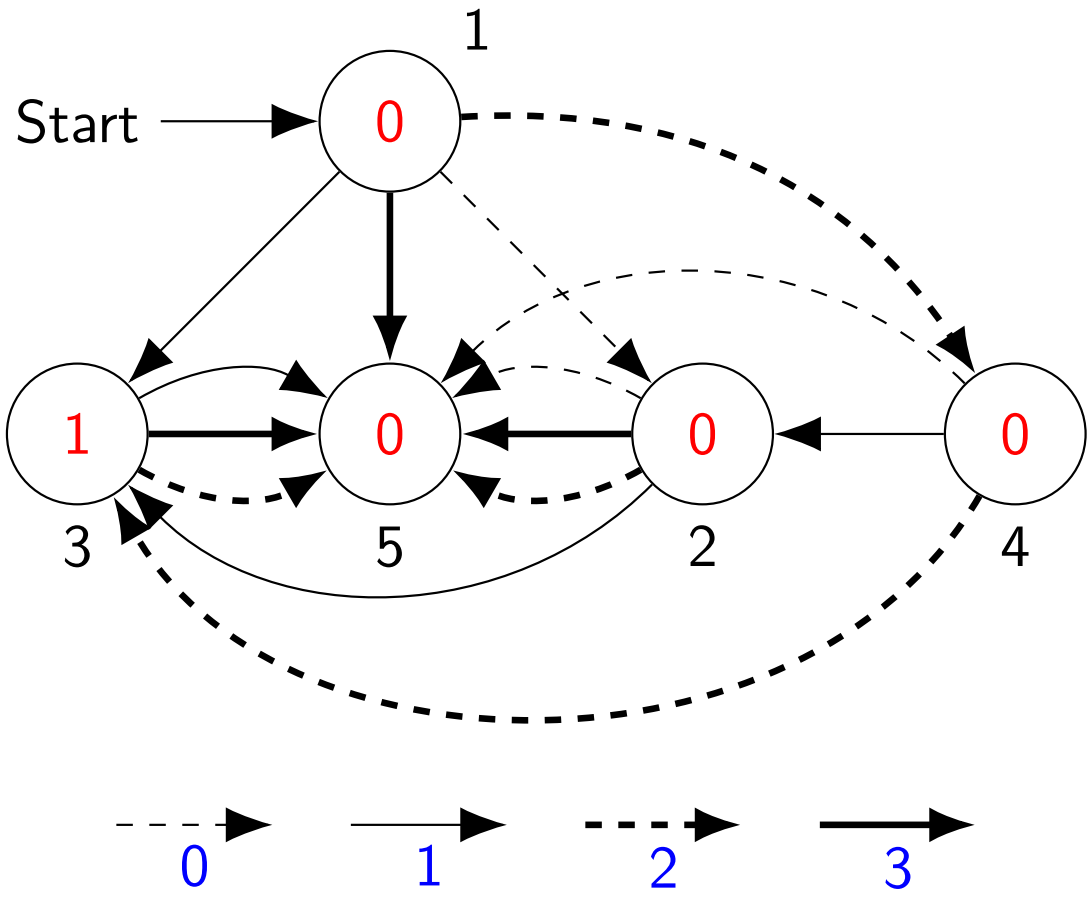


Automaton for σ_1^2

By similar reasoning we find that $\sigma_1^2(t)$ is a root of $f_{\sigma_1^2}(X) = t^2X^2 + X + t$.

We get the following automaton for σ_1^2:

State	0	1	2	3	σ_1^2 label
1	2	3	4	5	0
2	5	3	5	5	0
3	3	5	5	5	1
4	5	2	3	4	0
5	5	5	5	5	0



Computing automata for σ_1^3, σ_2^3, σ_3, and σ_3^3

Let $\sigma_3 = \sigma_1 \circ \sigma_2$. Automata for σ_1^3, σ_2^3, σ_3, and σ_3^3 can also be computed. We find that $\sigma_1^3(t)$, $\sigma_2^3(t)$, $\sigma_3(t)$, $\sigma_3^3(t)$ are roots of polynomials

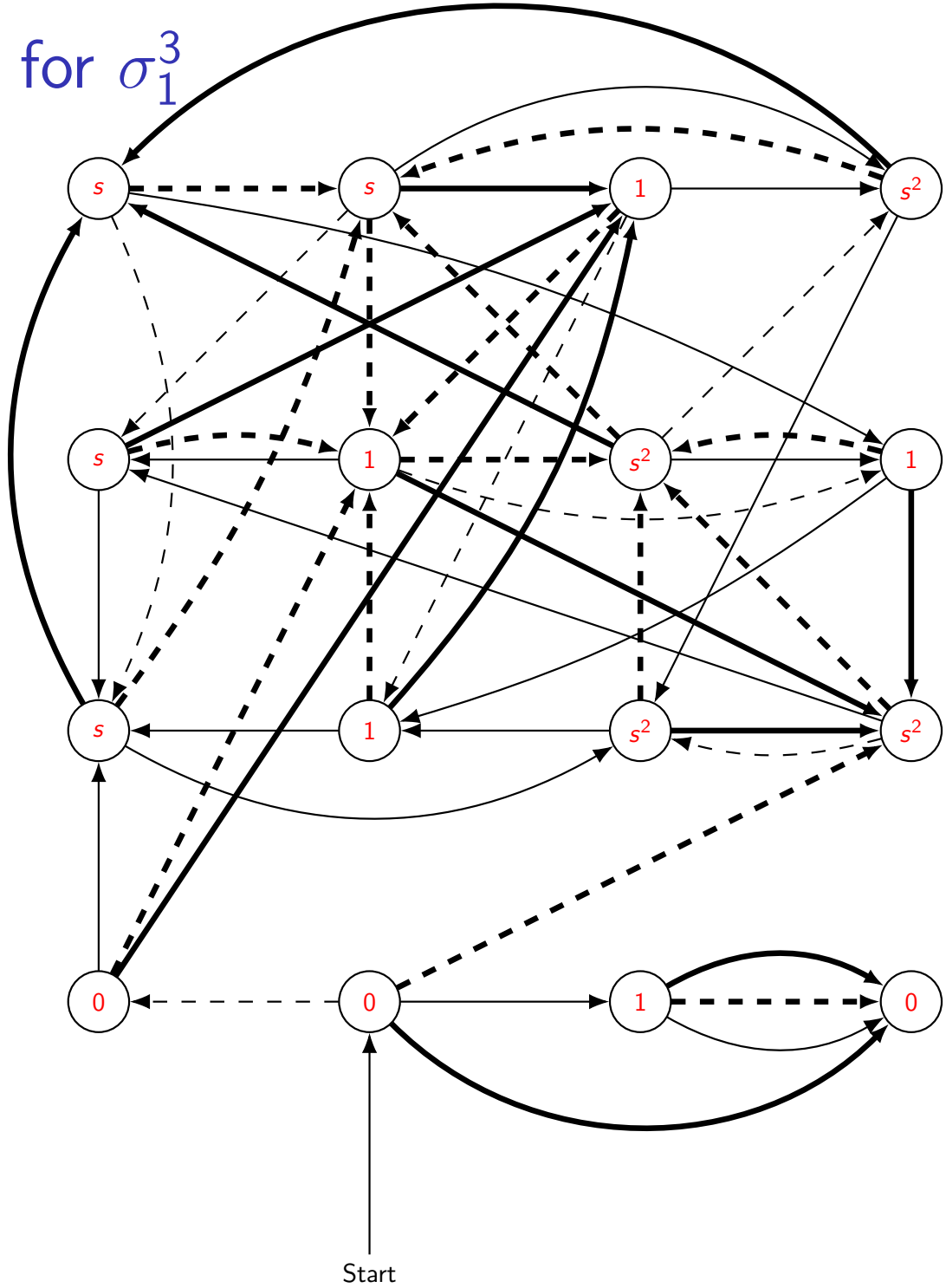
$$f_{\sigma_1^3}(t, X) = (t^2 + s)X^2 + X + t^2 + t$$
$$f_{\sigma_2^3}(t, X) = (t^2 + s^2)X^2 + X + t^2 + t$$
$$f_{\sigma_3}(t, X) = (t^2 + s^2)X^2 + X + st^2 + t$$
$$f_{\sigma_3^3}(t, X) = (t^2 + s)X^2 + X + s^2t^2 + t.$$

The automata for σ_1^3, σ_2^3, σ_3, and σ_3^3 all have the same digraph and the same edge labels, but different state labels:

Automata for σ_1^3, σ_2^3, σ_3, and σ_3^3

State	0	1	2	3	σ_1^3 label	σ_2^3 label	σ_3 label	σ_3^3 label
1	2	3	4	5	0	0	0	0
2	2	6	7	8	0	0	0	0
3	3	5	5	5	1	1	1	1
4	9	10	11	4	s^2	s	1	1
5	5	5	5	5	0	0	0	0
6	6	9	12	13	s	s^2	s	s^2
7	14	10	11	4	1	1	s	s^2
8	15	16	7	8	1	1	s^2	s
9	9	15	11	4	s^2	s	1	1
10	10	6	7	8	s	s^2	1	1
11	16	14	12	13	s^2	s	s^2	s
12	10	16	7	8	s	s^2	1	1
13	6	14	12	13	s	s^2	s	s^2
14	14	15	11	4	1	1	s	s^2
15	15	6	7	8	1	1	s^2	s
16	16	9	12	13	s^2	s	s^2	s

Automaton for σ_1^3



The case $\delta = s$

Suppose $\delta = s$, so that α_3 is a root of $X^2 - X - y^3 - s$.

In this case, Magma tells us that $\sigma_i(t)$ is a root of $f_{\sigma_i}(t, X)$, where

$$f_{\sigma_1}(t, X) = (1 + t + t^3)X^3 + (s^2 + st + t^2 + s^2t^3)X^2 + (1 + s^2t)X + t + s^2t^2 + t^3$$

$$f_{\sigma_2}(t, X) = (1 + t^2 + t^3)X^3 + (s^2 + st^2 + s^2t^3)X^2 + (1 + st + st^2 + st^3)X + t + s^2t^2 + t^3.$$

Applying the algorithms as above we find that σ_1, σ_2 are represented by automata with 175 and 169 states, respectively.

The other order-4 elements of this Q_8-subgroup of $\mathcal{N}(\mathbb{F}_4)$ are represented by automata with 500+ states.

D_4-subgroups of $\mathcal{N}(\mathbb{F}_4)$

We can use a similar approach to describe conjugacy classes of D_4-subgroups of $\mathcal{N}(\mathbb{F}_4)$ with minimum ramification breaks.

The smallest possible breaks for a D_4-subgroup of $\mathcal{N}(\mathbb{F}_4)$ are 1,1,5. There are three conjugacy classes of D_4-subgroups of $\mathcal{N}(\mathbb{F}_4)$ with these ramification breaks.

All three of these conjugacy classes contain a subgroup which is generated by two elements, each of which is represented by an automaton with 104 states.

For example . . .

Automatons for generators of a D_4-subgroup of $\mathcal{N}(\mathbb{F}_4)$

State	0	1	2	3	τ_1 label	τ_2 label	State	0	1	2	3	τ_1 label	τ_2 label
1	2	3	4	5	0	0	53	16	75	18	26	s	s^2
2	6	7	8	9	0	0	54	7	8	9	47	0	0
3	3	10	11	11	1	1	55	69	53	76	55	s^2	s
4	12	13	14	15	s^2	s	56	12	77	14	74	s^2	s
5	16	17	18	19	s	s^2	57	66	31	78	33	s	s^2
6	6	20	8	8	0	0	58	21	16	59	33	0	0
7	21	7	21	22	0	0	59	21	11	59	38	0	0
8	23	24	25	26	1	1	60	10	21	30	21	0	0
9	12	27	28	29	s^2	s	61	4	27	79	29	s^2	s
10	21	10	21	30	0	0	62	16	80	71	15	s	s^2
11	16	31	32	33	s	s^2	63	20	4	81	34	0	0
12	12	22	14	34	s^2	s	64	82	62	83	64	1	1
13	12	35	15	21	s^2	s	65	23	84	52	38	1	1
14	4	36	34	22	s^2	s	66	16	85	32	55	s	s^2
15	4	13	37	15	s^2	s	67	82	24	86	26	1	1
16	16	30	32	38	s	s^2	68	16	87	33	22	s	s^2
17	23	39	40	15	1	1	69	12	88	14	43	s^2	s
18	11	41	42	43	s	s^2	70	89	49	19	46	1	1
19	44	17	45	19	s	s^2	71	11	17	90	19	s	s^2
20	21	20	21	46	0	0	72	12	91	61	33	s^2	s
21	21	21	21	21	0	0	73	20	11	92	38	0	0
22	12	36	15	22	s^2	s	74	89	72	93	74	1	1
23	23	46	25	47	1	1	75	23	94	40	34	1	1
24	23	48	26	21	1	1	76	89	24	95	26	1	1
25	8	49	47	46	1	1	77	12	96	15	30	s^2	s
26	8	24	50	26	1	1	78	82	49	29	46	1	1
27	23	51	52	33	1	1	79	7	11	94	38	0	0
28	4	53	54	55	s^2	s	80	12	92	28	47	s^2	s
29	56	27	57	29	s^2	s	81	23	72	40	74	1	1

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