The ring of integers of degree p extensions of p-adic fields

Daniel Gil Muñoz

Charles University & Università di Pisa

Hopf Algebras & Galois Module Theory
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Notation

Let F be a p-adic field.

- $\mathcal{O}_F$ is the ring of integers of F.
- v_F is the valuation of F.
- π_F is a uniformizer of F.
- $\mathfrak{p}_F$ is the prime ideal of $\mathcal{O}_F$.
- $\overline{F}$ is the residue field of F.

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- $G_i := \{\sigma \in G \,|\, v_L(\sigma(\alpha) - \alpha) \geq i + 1 \text{ for all } \alpha \in \mathcal{O}_L\}$, $i \geq 0$.

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- **Ramification jump**: $t \in \mathbb{Z}_{\geq 1}$, $G_t \neq G_{t+1}$.

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Recall that

$$\mathfrak{A}_{L/K} = \{h \in H \mid h \cdot \mathcal{O}_L \subset \mathcal{O}_L\}.$$

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It is known that $p \mid t$ if and only if $t = \frac{rpe}{p-1}$.

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Complete characterization when L/K is Galois.

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Theorem

Let L/K be a totally ramified Galois degree p extension of p-adic fields. Let t be the ramification jump of L/K and $a = \mathrm{rem}(t, p)$.

1. *If $a = 0$, $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free.*
2. *If $a \mid p-1$, $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free. Moreover, if $t < \frac{pe}{p-1} - 1$, the converse holds.*
3. *If $t \geq \frac{pe}{p-1} - 1$, then $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free if and only if the length of the continued fraction expansion of $\frac{t}{p}$ is at most 4.*

Continued fraction expansion of $\frac{t}{p}$:

$$\frac{t}{p} = [a_0; a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

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Every rational number has a continued fraction expansion with finite length.

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Sketch of proof of the first two statements.

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Proof of the second statement (weaker version) using scaffolds.

- **I. Del Corso, F. Ferri, D. Lombardo (2022).**

Proof of the first two statements using the notion of minimal index.

G. (2023): Complete characterization when L/K has normal closure $\widetilde{L}$ dihedral of degree $2p$.

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Theorem

Let L/K be a degree p extension of p-adic fields with totally ramified dihedral degree $2p$ normal closure $\widetilde{L}$. Let t be the ramification jump of $\widetilde{L}/K$ and let $a = \mathrm{rem}(\ell := \frac{t+p}{2}, p)$.

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Let L/K be a degree p extension of p-adic fields with totally ramified normal closure $\widetilde{L}$. Let $r = [\widetilde{L} : L]$, t be the ramification jump of $\widetilde{L}/K$, let $c = \mathrm{rem}(t, r)$ and let $a = \mathrm{rem}(\ell := \frac{t+pc(r-1)}{r}, p)$.

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Let L' (resp. K') be the inertia field of $\widetilde{L}/L$ (resp. $\widetilde{L}/K$) and let $H' = K' \otimes_K H$.

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Proposition

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Then, we may assume without loss of generality that $\widetilde{L}/K$ is totally ramified.

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2. *If $a \mid p-1$, $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free. Moreover, if $t < \frac{rpe}{p-1} - r$, the converse holds.*
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- We prove that α is an eigenvector of the action of H, which allows us to obtain an $\mathcal{O}_K$-basis of $\mathfrak{A}_H$ of primitive pairwise orthogonal idempotents.
- We use this basis to establish an isomorphism $\varphi\colon H \longrightarrow K^p$ of K-algebras such that $\varphi(\mathfrak{A}_{L/K}) = \mathcal{O}_K^p$.

Therefore, $\mathfrak{A}_{L/K}$ is the maximal $\mathcal{O}_K$-order in $H \Longrightarrow$ $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free.

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G. Elder; *Ramified extensions of degree p and their Hopf-Galois module structure* J. Théor. Nr. Bordx. 1 (2018), 19-40.

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Main theorem (particular case): The typical degree p extensions of p-adic fields with totally ramified normal closure are the ones defined by an equation

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with $v_K(\alpha) = c$, $v_K(\beta) = -b$, where $b, c \in \mathbb{Z}$ are such that $0 \leq c < r$, $\gcd(c, r) = 1$ and $1 \leq bc + pr \leq \frac{rpe}{p-1}$.

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Moreover, the ramification jump of $\widetilde{L}/K$ is $t = br + pc$.

Since $p \nmid t$, L/K is typical. Let $x^p - \alpha^{\frac{p-1}{r}} x - \beta = 0$ be its defining equation.

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Theorem

Let L/K be a typical degree p extension of p-adic fields. The only Hopf-Galois structure on L/K is $H = K[w]$, where

$$w = y^{r-1}\Big(\sum_{m=1}^{p-1}\chi(m)^{-1}\sigma^m\Big).$$

Moreover, $w^p = \varepsilon p y^{(p-1)(r-1)}w$, $\varepsilon \in \mathcal{O}_K^*$.

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- $\chi \colon \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}_p$ is the p-adic Teichmuller character.

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Let L/K be a typical degree p extension of p-adic fields and let $H = K[w]$ be its only Hopf-Galois structure. Call $\ell = \frac{pc(r-1)+t}{r}$. Given $x \in L$,

$$v_L(w \cdot x) \geq \ell + v_L(x),$$

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with equality if and only if $p \nmid v_L(x)$.

In other words, w raises the valuations of elements at least ℓ, and exactly ℓ if and only if such a valuation is not divisible by p.

Given $\theta \in \mathcal{O}_L$, define

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The following statements are equivalent:

- $\mathcal{O}_L$ *is* $\mathfrak{A}_{L/K}$*-free.*
- $\mathfrak{A}_{L/K} = \mathfrak{A}_\theta$ *for some* $\theta \in \mathcal{O}_L$*.*
- $\mathfrak{A}_\theta$ *is a principal* $\mathfrak{A}_{L/K}$*-fractional ideal for some (any)* $\theta \in \mathcal{O}_L$*.*

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Thanks to w, we can find $\mathcal{O}_K$-bases of $\mathfrak{A}_{L/K}$ and $\mathfrak{A}_\theta$ with $\theta = \pi_L^a$.

2. If $a \mid p-1$, $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free. Moreover, if $t < \frac{rpe}{p-1} - r$, the converse holds.

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Roughly speaking, a scaffold on L/K consists in families of elements $\Psi_i \in H$ and $\lambda_j \in L$ such that the elements $\Psi_i \cdot \lambda_j$ have a prescribed valuation depending on i and j up to a certain precision $\mathfrak{c}$.

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- Weak requirement $\rightsquigarrow$ Sufficient condition for freeness.
- Strong requirement $\rightsquigarrow$ Necessary and sufficient condition for freeness.

Elder constructed a scaffold for L/K with precision

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Therefore, when $t < \frac{rpe}{p-1} - r$, $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free if and only if $a \mid p-1$.

3. If $t \geq \frac{rpe}{p-1} - r$, then $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free if and only if the length of the continued fraction expansion of $\frac{\ell}{p}$ is at most 4.

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Then $\mathfrak{A}_\theta$ is $\mathfrak{A}_{L/K}$-principal if and only if $\det(M(\alpha)) \not\equiv 0 \pmod{\mathfrak{p}_K}$ for some $\alpha \in \mathfrak{A}_\theta$.

We describe the entries of $M(\alpha)$ with the help of the set

$$E = \left\{ h \in \mathbb{Z} \,|\, 1 \leq h < p,\, 1 \leq h' < h \implies \widehat{h'\frac{a}{p}} > \widehat{h\frac{a}{p}} \right\},$$

which is parametrized in terms of the continued fraction expansion of $\frac{\ell}{p}$.

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- If $n \in \{3, 4\}$, we construct an α for which $\det(M(\alpha)) \not\equiv 0 \pmod{\mathfrak{p}_K}$, so $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free.

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- If $n \in \{3, 4\}$, we construct an α for which $\det(M(\alpha)) \not\equiv 0 \pmod{\mathfrak{p}_K}$, so $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free.
- If $n \geq 5$, we prove that $\det(M(\alpha)) \equiv 0 \pmod{\mathfrak{p}_K}$ for all $\alpha \in \mathfrak{A}_\theta$, so $\mathcal{O}_L$ is not $\mathfrak{A}_{L/K}$-free.

An interesting consequence

Proposition

If $G \cong C_p$ or D_p and $e = 1$, then $\mathcal{O}_L$ is $\mathfrak{A}_{L/K}$-free.

This does not necessarily hold in the general case.

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Example

Let L/K be a typical degree p extension with $p = 13$, $e = 1$, $r = 12$ and $t = 5$. Then $c = 5$ and $\ell = 60$. Now,

$$\frac{\ell}{p} = \frac{60}{13} = [4; 1, 1, 1, 1, 2].$$

Then $n = 5$ and $\mathcal{O}_L$ is not $\mathfrak{A}_{L/K}$-free.

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Thank you for your attention