A graph associated with finite skew braces

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NOTATIONS

A skew brace is a triple $(A, +, \circ)$, where $(A, +)$ and $(A, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

If $(A, +)$ is abelian, we call A a skew brace of abelian type. The λ-action is

$$\lambda : (A, \circ) \to \mathrm{Aut}(A, +) \qquad \lambda_a(b) = -a + a \circ b.$$

Let $b \in A$, the λ-orbit of b is $\Lambda(b) = \{\lambda_a(b) \colon a \in A\}$ and the stabilizer of b is $\mathrm{Stab}(b) = \{a \in A \colon \lambda_a(b) = b\}$.

$$\mathrm{Fix}(A) = \{b \in A \colon \lambda_a(b) = b \ \forall a \in A\}$$

is the additive subgroup of the trivial λ-orbits.

DEFINITION

Definition (Bertram-Herzog-Mann)

For a finite group G, let $\Gamma(G)$ is the graph with vertices the non-trivial conjugacy classes of G and two vertices C_1, C_2 are adjacent if $\gcd(|C_1|, |C_2|) \neq 1$.

DEFINITION

Definition (for skew braces)

For a finite skew brace A, let $\Gamma(A)$ is the graph with vertices the non-trivial λ-orbits of A and two vertices C_1, C_2 are adjacent if $\gcd(|C_1|, |C_2|) \neq 1$.

Connection:

If $(G, \cdot)$ is a finite group, then $\Gamma(G, \cdot, \cdot^{op}) = \Gamma(G)$: on the skew brace $(G, \cdot, \cdot^{op})$, the λ-action is

$$\lambda_g(h) = g^{-1} \cdot (h \cdot^{op} g) = g^{-1}hg.$$

EXAMPLES

Let $(A, +, \circ)$ be a finite skew brace.

- $\Gamma(A)$ has no vertices if and only if $+ = \circ$.
- If $|A| = p^2$, then $\Gamma(A)$ is empty or a complete graph with $p - 1$ vertices.
- If $|A| = pq$, then $\Gamma(A)$ is completely determined by $|\operatorname{Fix}(A)|$.

EXAMPLES

$(A,+)$	$(n,m)\circ(s,t)$	$\lvert \mathrm{Fix}(A) \rvert$	$\Gamma(A)$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(n+s, m+t)$	6	
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$(n+(-1)^m s, m+t)$	6	
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(n+(-1)^m s, m+t)$	2	•—•
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$((-1)^t n+(-1)^m s, m+t)$	3	•
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$(n+s, m+t)$	2	•—•
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$((-1)^t n+s, m+t)$	1	• •

Table: Skew braces of size 6 [Acri-Bonatto].

PROPERTIES

Proposition

If A is a finite skew brace, then the number of connected components of $\Gamma(A)$ is

$$n(\Gamma(A)) \leq 2.$$

Proposition

If A is a finite skew brace such that $n(\Gamma(A)) = 1$, then the diameter of $\Gamma(A)$ is

$$d(\Gamma(A)) \leq 4.$$

TWO DISCONNECTED VETICES

Theorem

Let A be a finite skew brace. If $\Gamma(A)$ has exactly two disconnected vertices, then $A \cong (\mathcal{S}_3, \cdot, \cdot^{\mathrm{op}})$.

$(A, +)$	$(n, m) \circ (s, t)$	$\lvert \mathrm{Fix}(A) \rvert$	$\Gamma(A)$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(n + s, m + t)$	6	
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$(n + (-1)^m s, m + t)$	6	
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(n + (-1)^m s, m + t)$	2	•—•
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$((-1)^t n + (-1)^m s, m + t)$	3	•
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$(n + s, m + t)$	2	•—•
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$((-1)^t n + s, m + t)$	1	• •

Table: Skew braces of size 6 [Acri-Bonatto].

ONE VERTEX: ABELIAN TYPE

Theorem

Let A be a finite skew brace of abelian type such that $\Gamma(A)$ has only one vertex. Then A is isomorphic to one of the following skew braces.

- *On $\mathbb{Z}/4\mathbb{Z}$, with multiplication $x \circ y = x + y + 2xy$.*
- *On $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, with multiplication*

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 + x_2 + y_1 y_2, y_1 + y_2).$$

- *On $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, with multiplication*

$$(x_1, y_1) \circ (x_2, y_2) = \left(x_1 + x_2 + y_2 \sum_{i=1}^{y_1 - 1} i, y_1 + y_2 + 2y_1 y_2 \right).$$

ONE VERTEX: ABELIAN TYPE

Sketch of the proof:

$A = \mathrm{Fix}(A) \sqcup \Lambda(x)$ for some $x \in A$.

- $|\Lambda(x)| = |A|/2 = |\mathrm{Fix}(A)|$ and $|\ker \lambda| = 2$.
- $\mathrm{Fix}(A)$ is abelian (and A is left nilpotent).
- There is no decomposition $A = A_1 \times A_2$.
- $|A| = 2^m$
  ([Cedó-Smoktunowicz-Vendramin] decomposition).
- $|A| \leq 8$:
  - $|A| = 4 \iff (A, \circ)$ is abelian.
  - If $|A| > 4$. Consider $\overline{A} = A/\ker \lambda$:

    $\Gamma(\overline{A}) = \bullet$ and $(\overline{A}, \circ) \cong \mathrm{Fix}(A)$ abelian $\Rightarrow |\overline{A}| = 4$.

ONE VERTEX: GENERAL CASE

Theorem

Let A be a finite skew brace. $\Gamma(A)$ has exactly one vertex if and only if A is isomorphic to a skew brace on the set $F \times \mathbb{Z}/2\mathbb{Z}$, with

$$(f_1, k_1) + (f_2, k_2) = \big(f_1 + (-1)^{k_1} f_2 + k_1 k_2 y, k_1 + k_2\big),$$
$$(f_1, k_1) \circ (f_2, k_2) = \big(f_1 + \psi(f_1, k_1, k_2) + (-1)^{k_1} f_2 + k_1 k_2 y, k_1 + k_2\big),$$

where $F \neq \{0\}$ is an abelian group, $y \in F$ such that $2y = 0$, and $\psi \colon F \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to F$ is a surjective map such that

$$\psi(f_1, k_1, k_2) = \frac{1 - (-1)^{k_2}}{2} \left(\phi(f_1) - \frac{1 - (-1)^{k_1}}{2} z \right),$$

where $\phi \in \mathrm{End}(F)$, $z \in F$, $\phi(z) = \phi(y) - 2z$, and $\phi^2 = -2\phi$.

ONE VERTEX: GENERAL CASE

With these conditions there exists an abelian group G of odd order such that

$$F \cong (\mathbb{Z}/2\mathbb{Z}/ \times \mathbb{Z}/2\mathbb{Z}) \times G \text{ and } \phi = (\alpha, -2\,\mathrm{id}_G) \text{ with } |\ker \alpha| = 2,$$

or

$$F \cong \mathbb{Z}/2^i\mathbb{Z} \times G \text{ and } \phi = -2\,\mathrm{id}_F\,.$$

Corollary

The number of isomorphism classes of skew braces A with one-vertex graph $\Gamma(A)$ of size $n = 2^m d$, for $\gcd(2, d) = 1$ is

$$\begin{cases} m \cdot Ab(d) & \text{if } 0 \leq m \leq 3, \\ 2 \cdot Ab(d) & \text{if } m \geq 4, \end{cases}$$

where $Ab(d)$ is the number of abelian groups of order d.

QUESTIONS

- Can we characterize skew braces with a graph with two connected components?
  (for groups in [Bertram-Herzog-Mann]: quasi-Frobenius with abelian kernel and complement)

- Is it true (as it is for groups, [Chillag-Herzog-Mann]) that in the connected case, $d(\Gamma(A)) \leq 3$?

- When does $d(\Gamma(A)) \leq 2$?

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