The structure monoid of set-theoretic solutions to the YBE

Ilaria Colazzo

I.Colazzo@exeter.ac.uk

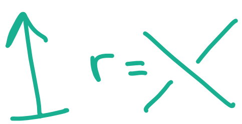
April 18, 2023

InterplaySbHG – The Interplay Between Skew Braces and Hopf-Galois Theory

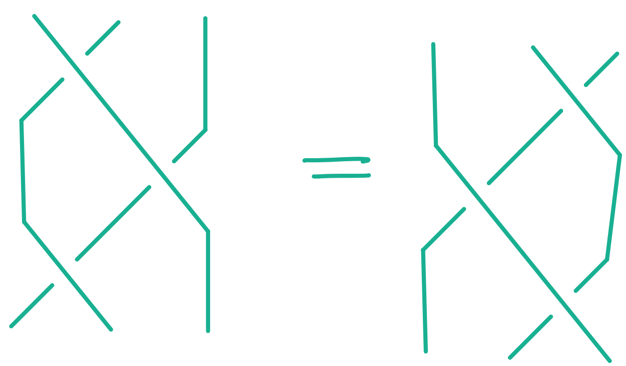
Set-theoretic solutions to the Yang-Baxter equation

A set-theoretic solution (to the YBE) is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$$



YBE



Set-theoretic solutions to the Yang-Baxter equation

Convention: If (X, r) is a set-theoretic solution to the YBE, we write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

- (X, r) is bijective if r is bijective.
- (X, r) is finite if X is finite.
- (X, r) is left non-degenerate (resp. right non-degenerate) if λ_x (resp. ρ_x) are bijective for all $x \in X$.
- (X, r) is non-degenerate if it is both left and right non-degenerate.

Examples

X a set.

- $r(x,y)=(x,y)$ is a degenerate involutive solution.
- $r(x,y)=(y,x)$ is an involutive non-degenerate solution.
- $r(x,y)=(y,y)$ is a left non-degenerate solution.
- f,g maps. $r(x,y)=(f(y),g(x))$ is a solution if and only if $fg=gf$.
  - r is bijective if and only if $f,g\in \mathrm{Sym}(X)$.
    (in such a case r is also non-degenerate)
  - r is involutive if and only if $f=g^{-1}$.

Examples

X a set.

- $r(x, y) = (x, y)$ is a degenerate involutive solution.
- $r(x, y) = (y, x)$ is an involutive non-degenerate solution.
- $r(x, y) = (y, y)$ is a left non-degenerate solution.
- f, g maps. $r(x, y) = (f(y), g(x))$ is a solution if and only if $fg = gf$.
  (X, r) is called a permutational solution or a Lyubashenko's solution.

Examples

G a group

- $r(x,y) = (xy, 1)$ is a left non-degenerate solution.
- $r(x,y) = (y, y^{-1}xy)$ is a bijective non-degenerate solution.
- $r(x,y) = (x^2y, y^{-1}x^{-1}y)$ is a bijective non-degenerate solution.

The structure group

Definition. Let (X, r) be a solution. Define the structure group

$$G(X,r) = \mathrm{gr}(X \mid x \circ y = \lambda_x(y) \circ \rho_y(x)).$$

Theorem. (Etingof, Schedler and Soloviev, 1999). Let (X, r) be an involutive non-degenerate solution, then

$$G(X,r) \overset{\text{regular}}{\hookrightarrow} \mathbb{Z}^{|X|} \rtimes \operatorname{Sym} X.$$

Moreover, (X, r) can be extended to a solution on $G(X, r)$.

Bijective non-degenerate solutions

Theorem (Lu, Yan and Zhu, 2000, Soloviev 2000, Lebed and Vendramin 2017). Let (X, r) be an bijective non-degenerate solution, then

$$G(X, r) \overset{\text{regular}}{\hookrightarrow} A_g(X, r) \rtimes \operatorname{Sym} X,$$

where

$$A_g(X, r) = \operatorname{gr}(X \mid x + \lambda_x(y) = \lambda_x(y) + \lambda_{\lambda_x(y)}\rho_y(x))$$
$$= \operatorname{gr}(X \mid x + z = z + \sigma_z(x)).$$

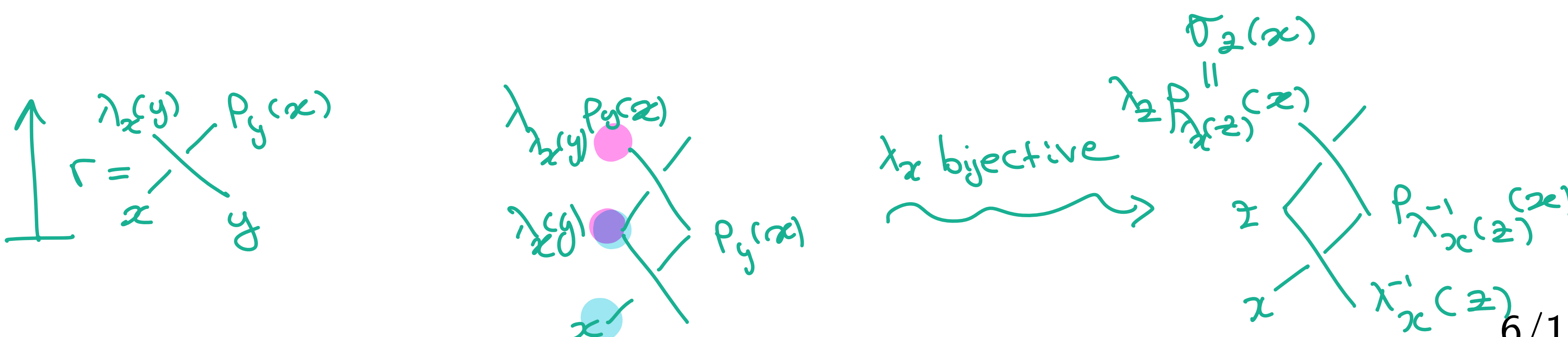
Bijective non-degenerate solutions

Theorem (Lu, Yan and Zhu, 2000, Soloviev 2000, Lebed and Vendramin 2017). Let (X, r) be an bijective non-degenerate solution, then

$$G(X,r) \overset{\text{regular}}{\hookrightarrow} A_g(X,r) \rtimes \operatorname{Sym} X,$$

where

$$\begin{aligned} A_g(X,r) &= \operatorname{gr}(X \mid x + \lambda_x(y) = \lambda_x(y) + \lambda_{\lambda_x(y)}\rho_y(x)) \\ &= \operatorname{gr}(X \mid x + z = z + \sigma_z(x)). \end{aligned}$$



An "extension" of (X,r)

Theorem (Lu, Yan and Zhu, 2000, Smoktunowicz and Vendramin, 2018). If (X, r) is a bijective non-degenerate solution, then there exists a unique solution on $G(X, r)$ such that

$$r_{G(X,r)}(\iota \times \iota) = (\iota \times \iota)r$$

where $\iota : X \to G(X, r)$ is the canonical map.

The main issue of this correspondence is that ι is not an injective map in general.

Example. Let $X = \{1, 2, 3, 4\}$ be a set, $f = (1\ 2)$ and $g = (3\ 4)$, then $fg = gf$ and the map $r(x, y) = (f(y), g(x))$ is a solution to the YBE.
It is easy to see that (X, r) is not injective. Indeed in $G(X, r)$ we have $1 = 2$ and $3 = 4$.

Injective solutions

A bijective non-degenerate solution is injective if the map $\iota : X \to G(X, r)$ is injective.

Examples.

1. Solutions associated to skew braces are injective.
2. Irretractable solutions are injective.

The retraction

Let (X, r) be a bijective non-degenerate solution and define on X the following relation

$$x \sim y \quad \Longleftrightarrow \quad \lambda_x = \lambda_y \text{ and } \rho_x = \rho_y.$$

Let $\bar{X} = X/\sim$ denote the set of equivalence classes and let $[x]$ denote the class of x.
Then r induce a solution $\bar{r}$ on $\bar{X}$ by

$$\bar{r}([x],[y]) = ([\lambda_x(y)],[\rho_y(x)]),$$

for all $x, y \in X$.
The solution $\mathrm{Ret}(X, r) = (\bar{X}, \bar{r})$ is called the retraction of (X, r).

The injectivization

Let (X, r) be a bijective non-degenerate solution and let $\iota : X \to G(X, r)$ be the canonical map. Then

$$\mathrm{Inj}(X, r) = (\iota(X), r_{G(X,r)}|_{\iota(X)\times\iota(X)})$$

is a bijective non-degenerate solution and

$$G(X, r) \cong G(\iota(X), r_{G(X,r)}|_{\iota(X)\times\iota(X)}).$$

Left-non-degenerate solutions

Let (X, r) be a left non-degenerate solution.
The structure monoid is the monoid

$$M(X, r) = \langle X \mid x \circ y = \lambda_x(y) \circ \rho_y(x) \rangle.$$

Theorem (Gateva-Ivanova and Majid, 2005, Cedó, Jespers and Verwimp, 2021).

$$M(X, r) \overset{\text{regular}}{\hookrightarrow} A(X, r) \rtimes \operatorname{Sym} X,$$

where

$$\begin{aligned} A(X, r) &= \langle X \mid x + \lambda_x(y) = \lambda_x(y) + \lambda_{\lambda_x(y)\rho_y(x)} \rangle \\ &= \langle X \mid x + z = z + \sigma_z(x) \rangle. \end{aligned}$$

An “extension” of (X,r)

Theorem (Gateva-Ivanova and Majid, 2005, IC, Jespers, Van Antwerpen and Verwimp, 2022). If (X, r) is a left non-degenerate solution, then there exists a unique solution on $M(X, r)$ such that

$$r_{M(X,r)}(\iota \times \iota) = (\iota \times \iota)r$$

where $\iota : X \to G(X, r)$ is the canonical map.

In this case, the canonical map is an embedding.

Question

Let (X, r) be a left non-degenerate solution. What is the relation between (X, r) being right non-degenerate and r being bijective?

- Any finite involutive left non-degenerate solution is non-degenerate.

  [Rump, 2005]

  [Jespers and Okniński, 2005]

- An example of an infinite involutive solution that is left non-degenerate but not right non-degenerate.

  [Rump, 2005]

- Any non-degenerate solution such that $\lambda_x = \lambda_y$ implies $x = y$ is bijective.

  [Cedó, Jespers and Verwimp, 2021]

- Any finite bijective left non-degenerate solution is right non-degenerate

  [Castelli, Catino, Stefanelli, 2021]

Theorem

If (X, r) is a finite left non-degenerate solution, then

$$r \text{ is bijective} \iff (X, r) \text{ is right non-degenerate}$$

- $\Rightarrow$ Castelli, Catino and Stefanelli, 2021.
- $\Leftarrow$ IC, Jespers, Van Antwerpen, Verwimp, 2022.

The cancellative congruence

Theorem (IC, Jespers, Kubat, Van Antwerpen, 2023).
Assume (X, r) is a left non-degenerate solution of the YBE. If η_A denotes the cancellative congruence of $A = A(X, r)$ and η_M denotes the cancellative congruence of $M = M(X, r) = \{(a, \lambda_a) : a \in A\}$ then

$$\eta_A = \{(a, b) \in A \times A : a + c = b + c \text{ for some } c \in A\}$$

and

$$\begin{aligned}\eta_M &= \{(x, y) \in M \times M : x \circ z = y \circ z \text{ for some } z \in M\} \\ &= \{((a, \lambda_a), (b, \lambda_b)) \in M \times M : (a, b) \in \eta_A \text{ and } \lambda_a = \lambda_b\}.\end{aligned}$$

Let $x, y \in X$. As a consequence, we have that

$$(x, y) \in \eta_M \Longrightarrow x \sim_{\mathrm{Ret}} y.$$

Hence,

$$(X, r) \longrightarrow \mathrm{Inj}(X, r) \longrightarrow \mathrm{Ret}(X, r).$$

An Application.

Theorem (IC, Van Antwerpen, in progress) Let (X, r) be a bijective non-degenerate solution. Then (X, r) is decomposable if and only if $\mathrm{Inj}(X, r)$ is decomposable.

A solution (X, r) is said to be decomposable if there exist $Y, Z \subseteq X$ such that $Y, Z \neq$, $Y \cap Z =$, $Y \cup Z = X$, $(Y, r_{|_{Y\times Y}})$ and $(Z, r_{|_{Z\times Z}})$ are solutions.