Canonical ideals in skew bracoids

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Overview

- Joint work with Isabel Martin-Lyons

Aim

Formulate skew bracoid analogues of important "canonical" ideals in skew braces. Begin to explore chains of ideals, pointing towards notions of nilpotency etc.

- Reminder of definition of skew bracoids and connection with Hopf-Galois theory
- γ-functions, ideals, associated subgroups
- Some canonical ideals: centre, socle, $*$ operation
- Chains of ideals

Skew bracoids

Definition

A *skew bracoid* is a 5-tuple $(G, \cdot, N, \star, \odot)$ where $(G, \cdot)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of $(G, \cdot)$ on N such that

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot e_N)^{-1} \star (g \odot \mu)$$

for all $g \in G$ and $\eta, \mu \in N$.

- Where possible, write $(G, N, \odot)$, or even (G, N).
- Where possible, write $g \cdot h = gh$ and $\eta \star \mu = \eta\mu$.
- Every skew brace is a skew bracoid, with $\odot$ and $\cdot$ coinciding.
- If $\mathrm{Stab}_G(e_N) = \{e_G\}$ then (G, N) is essentially a skew brace.

An example

Example

- Let $(A, \star, \cdot)$ be a skew brace and let B be a strong left ideal.
- B is a normal subgroup of $(A, \star)$, so $(A/B, \star)$ is a group.
- B is a subgroup of $(A, \cdot)$, and the cosets of B with respect to $\cdot$ and $\star$ coincide.
- $(A, \cdot)$ acts by left translation on the coset space A/B. Write $\odot$ for this action.
- Then $(A, \cdot, A/B, \star, \odot)$ is a skew bracoid.

Question

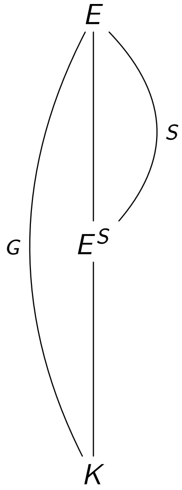
Does every skew bracoid occur in this way?

Connection with Hopf-Galois theory

Theorem

Let E/K be a finite Galois extension with Galois group $(G, \cdot)$, and let $S \leq G$. There are bijections between

- *binary operations $\star$ on $X = G/S$ such that $(G, \cdot, X, \star, \odot)$ is a skew bracoid;*
- *regular G-stable subgroups of $\mathrm{Perm}(X)$;*
- *Hopf-Galois structures on E^S/K.*



- Nice theorems of Stefanello/Trapenniers generalize to this setting; not the focus of this talk.

Reduction and equivalence

Proposition

If $(G, N, \odot)$ is a skew bracoid then the image of the permutation representation $\lambda_\odot : G \to \mathrm{Perm}(N)$ is contained in $\mathrm{Hol}(N) \cong N \rtimes \mathrm{Aut}(N)$.

Definition

Two skew bracoids (G, N) and (G', N') are called *equivalent* if $N = N'$ and $\lambda_\odot(G) = \lambda_{\odot'}(G') \subseteq \mathrm{Hol}(N)$.

Proposition

Let (G, N) be a skew bracoid.

- Let $\overline{G} = G/\ker(\lambda_\odot)$. Then $(\overline{G}, N)$ is skew bracoid in which $\overline{G}$ acts faithfully on N. This is called the *reduced form* of (G, N).
- Every skew bracoid is equivalent to its reduced form.

γ-functions

Proposition

Let (G, N) be a skew bracoid. Define $\gamma : G \to \mathrm{Perm}(N)$ by

$$ {}^{\gamma(g)}\eta = (g \odot e_N)^{-1} \star (g \odot \eta) \text{ for all } g \in G \text{ and } \eta \in N. $$

Then

- γ is a group homomorphism;
- $\gamma(G) \subseteq \mathrm{Aut}(N)$.

The function γ is called the *γ-function* of the skew bracoid (G, N).

What if γ is trivial?

Proposition

Suppose that (G, N) is a skew bracoid such that ${}^{\gamma(g)}\eta = \eta$ for all $g \in G$ and all $\eta \in N$. Then $(\overline{G}, N)$ is essentially a trivial skew brace.

Proof.

For all $g \in G$ and $\eta \in N$ we have

$$\begin{aligned} & (g \odot e_N)^{-1}(g \odot \eta) = \eta \\ \Rightarrow \quad & (g \odot \eta) = (g \odot e_N)\eta. \end{aligned}$$

Hence $\ker(\lambda_\odot) = \mathrm{Stab}_G(e_N)$, so $\mathrm{Stab}_{\overline{G}}(e_N) = \{\overline{e_G}\}$, so $(\overline{G}, N)$ is essentially a skew brace.

The γ-function of $(\overline{G}, N)$ is given by ${}^{\gamma(\overline{g})}\eta = {}^{\gamma(g)}\eta = \eta$, so $(\overline{G}, N)$ is essentially a trivial skew brace. □

Ideals

Let (G, N) be a skew bracoid.

Definition

A *left ideal* of (G, N) is a subgroup M of N such that ${}^{\gamma(G)}M = M$.

An *ideal* of (G, N) is a left ideal that is normal in N.

Proposition

If M is an ideal of (G, N) then $(G, N/M)$ is a skew bracoid.

Definition

The *associated subgroup* of a left ideal M is

$$G_M = \{g \in G \mid g \odot \mu \in M \text{ for all } \mu \in M\} \leq G.$$

We call M an *enhanced* left ideal if G_M is normal in G.

Canonical ideals: characteristic subgroups

Proposition

Let (G, N) be a skew bracoid and suppose that M is a characteristic subgroup of N. Then N is an ideal of (G, N).

Proof.

Clearly M is normal in N, and since $\gamma(G) \subset \mathrm{Aut}(N)$ we have ${}^{\gamma(G)}M = M$. □

A natural example is $M = Z(N)$, of course.

Question

Does the associated subgroup of $Z(N)$ have a nice characterization?

Canonical ideals: socle

- If $(A, \star, \cdot)$ is a skew brace then $\mathrm{Soc}(A) = \ker(\gamma) \cap Z(A, \star)$ is an ideal.

Proposition

If (G, N) is a skew bracoid then $\mathrm{Soc}(G, N) = (\ker(\gamma) \odot e_N) \cap Z(N)$ is an ideal of (G, N).

Proof.

For $k \in \ker(\gamma)$ and $h \in G$ we have

$$\begin{aligned} (k \odot e_N)(h \odot e_N) &= (k \odot e_N)^{\gamma(k)}(h \odot e_N) \\ &= (k \odot e_N)(k \odot e_N)^{-1}(k \odot (h \odot e_N)) \\ &= ((kh) \odot e_N). \end{aligned}$$

Hence $(\ker(\gamma) \odot e_N) \leq N$, and so $\mathrm{Soc}(G, N) \trianglelefteq N$.

Canonical ideals: socle

Recall: $\mathrm{Soc}(G,N) = (\ker(\gamma) \odot e_N) \cap Z(N) \trianglelefteq N$.

Remains to show that $\mathrm{Soc}(G,N)$ is $\gamma(G)$-stable.

Proof continued...

Let $\mu \in \mathrm{Soc}(G,N)$ and write $\mu = k \odot e_N$ with $k \in \ker(\gamma)$.

Let $g \in G$. Then ${}^{\gamma(g)}\mu \in Z(N)$ and

$$\begin{aligned}
{}^{\gamma(g)}\mu &= (g \odot e_N)^{-1}(g \odot (k \odot e_N)) \\
&= g \odot ((g^{-1} \odot e_N)(k \odot e_N)) \\
&= g \odot ((k \odot e_N)(g^{-1} \odot e_N)) \\
&= g \odot (kg^{-1} \odot e_N) \\
&= gkg^{-1} \odot e_N \in \ker(\gamma) \odot e_N.
\end{aligned}$$

Hence $\mathrm{Soc}(G,N)$ is an ideal of (G,N). □

The $*$ operation

- If $(A, \star, \cdot)$ is a skew brace then $* : A \times A \to A$ is defined by

$$a * b = a^{-1} \star (a \cdot b) \star b^{-1} = {}^{\gamma(a)}(b) \star b^{-1}.$$

- A subgroup B of $(A, \star)$ is a left ideal of A if and only if $A * B \subset B$.

Definition

Let (G, N) be a skew bracoid. Define $* : G \times N \to N$ by $g * \eta = {}^{\gamma(g)}(\eta)\eta^{-1}$.

Proposition

A subgroup M of N is a left ideal of (G, N) if and only if $G * M \subset M$.

Left series

- If A is a skew brace then let $A^1 = A$ and for $i \geq 1$ let
$$A^{i+1} = A * A^i = \langle a * b \mid a \in A,\ b \in A^i \rangle_\star$$
Then the A^i form a descending chain of left ideals of A.

Proposition

Let (G, N) be a skew bracoid. Let $N^1 = N$ and for $i \geq 1$ let

$$N^{i+1} = G * N^i = \langle g * \eta \mid g \in G,\ \eta \in N^i \rangle.$$

Then the N^i form a descending chain of left ideals of (G, N).

Proof.

We have $N^i \leq N$ for each i by definition.

Suppose that N^i is a left ideal.

We must show that ${}^{\gamma(h)}(g * \eta) \in N^{i+1}$ for all $g, h \in G$ and $\eta \in N^i$.

Left series

Recall: $N^{i+1} = G * N^i = \langle g * \eta \mid g \in G,\ \eta \in N^i \rangle$.

We must show that ${}^{\gamma(h)}(g * \eta) \in N^{i+1}$ for all $g, h \in G$ and $\eta \in N^i$.

Proof continued...

We have:

$$\begin{aligned}
{}^{\gamma(h)}(g * \eta) &= {}^{\gamma(h)}\left({}^{\gamma(g)}(\eta)\eta^{-1}\right) \\
&= {}^{\gamma(h)\gamma(g)}(\eta)\,{}^{\gamma(h)}(\eta^{-1}) \\
&= {}^{\gamma(hgh^{-1})}\left({}^{\gamma(h)}(\eta)\right){}^{\gamma(h)}(\eta)^{-1} \\
&= hgh^{-1} * \left({}^{\gamma(h)}(\eta)\right) \\
&\in G * N^i.
\end{aligned}$$

Hence N^{i+1} is a left ideal of (G, N). □

N^2 is special

- If A is a skew brace then $A^2 = A * A$ is an ideal and A/A^2 is a trivial skew brace.

Proposition

Let (G, N) be a skew bracoid. Then $N^2 = G * N$ is an ideal of (G, N).

Proof.

We need to show that $N^2 \trianglelefteq N$.

It is easy to show that for all $g \in G$ and $\mu, \eta \in N$ we have

$$g * (\mu\eta) = (g * \mu)\mu(g * \eta)\mu^{-1}.$$

Hence

$$\mu(g * \eta)\mu^{-1} = (g * \mu)^{-1}(g * (\mu\eta)) \in G * N.$$

□

N^2 is special

Proposition

Let (G, N) be a skew bracoid. Then the reduced form of $(G, N/N^2)$ is essentially a trivial skew brace.

Proof.

The γ-function of $(G, N/N^2)$ is given by ${}^{\gamma(g)}(\eta N^2) = ({}^{\gamma(g)}\eta)N^2$.
We have

$$\begin{aligned} & g * \eta = {}^{\gamma(g)}(\eta)\eta^{-1} \in N^2 \trianglelefteq N \\ \Rightarrow \quad & \left(\eta^{-1}\right) {}^{\gamma(g)}(\eta) \in N^2 \\ \Rightarrow \quad & ({}^{\gamma(g)}\eta)N^2 = \eta N^2. \end{aligned}$$

Thus the γ-function of $(G, N/N^2)$ is trivial, and so $(\overline{G}, N/N^2)$ is essentially a trivial skew brace. □

Right series?

- If A is a skew brace then let $A^{(1)} = A$ and for $i \geq 1$ let
$$A^{(i+1)} = A^{(i)} * A = \langle b * a \mid b \in A^{(i)},\ a \in A \rangle_{\star}$$
Then the $A^{(i)}$ form a descending chain of ideals of A.

A possible route for generalizing this to a skew bracoid (G, N) might be:

- Let $N^{(1)} = N$.
- Let $N^{(2)} = G * N$.
  This is an ideal; let $G_{(2)}$ denote its associated subgroup.
- Let $N^{(3)} = G_{(2)} * N$.
  Assuming this is at least a left ideal, let $G_{(3)}$ denote its associated subgroup.
- etc.

In order to make sense of this, we need to understand more about the associated subgroups $G_{(i)}$.

Further questions

- Could left/right series lead to notions of left/right nilpotency?
- What about solubility?
- What are the consequences of these notions
  - in Hopf-Galois theory?
  - in the construction/classification of skew bracoids?
  - for solutions of the Yang-Baxter equation?

Thank you for your attention.