Skew bracoid webs arising from abelian maps

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Outline

Bracoids

Recall: a skew left bracoid (hereafter, bracoid) is a quintuple $(G, \cdot, N, \star, \odot)$ such that:

1. $(G, \cdot)$ and $(N, \star)$ are groups;
2. G acts on N via $(g, \eta) \mapsto g \odot \eta, \; g \in G, \eta \in N$; and
3. the following bracoid relation holds:

$$g \odot (\eta \star \pi) = (g \odot \eta) \star (g \odot 1_N)^{-1} \star (g \odot \pi), \; g \in G, \; \eta, \pi \in N.$$

Conventions:

- Write (G, N) when operations are understood.
- Write gh for $g \cdot h$.
- Write $(\;)^{-1}$ for the inverse in G and N.

$$g \odot (\eta \star \pi) = (g \odot \eta) \star (g \odot 1_N)^{-1} \star (g \odot \pi)$$

Examples

1. A skew left brace $(B, \circ, \cdot)$ is a bracoid: $G = (B, \cdot)$, $N = (B, \circ)$ and $a \odot b = a \cdot b$. Note that $(B, \circ)$ is the "additive group".
2. Let $G = S_3$, $N = C_3 = \langle \eta \rangle$. Define $\odot$ by

g	$g \odot \eta^i$
ι	η^i
(12)	η^{-i}
(13)	η^{1-i}
(23)	η^{2-i}
(123)	η^{i+1}
(132)	η^{i+2}

.

Then (S_3, C_3) is a bracoid.

Abelian maps and braces

Let $G = (G, \cdot)$ be a group, usually taken to be nonabelian.

We say $\psi \in \mathrm{End}(G)$ is an abelian map if $\psi(G)$ is abelian.

Denote the set of abelian maps on G by $\mathrm{Ab}(G)$.

$\psi \in \mathrm{Ab}(G)$ gives rise to a biskew brace $(G, \circ, \cdot)$, where the new operation is given by

$$g \circ h = g\psi(g^{-1})h\psi(g), \; g, h \in G.$$

Note that $\psi_1, \psi_2 \in \mathrm{Ab}(G)$ give the same binary operation iff $\psi_1(g)\psi_2(g)^{-1} \in Z(G)$ for all $g \in G$.

$$g \circ h = g\psi(g^{-1})h\psi(g)$$

Note, for example, that the trivial map yields the trivial brace.

While $(G, \circ)$ is not necessarily isomorphic to $(G, \cdot)$, there is a homomorphism $\phi : (G, \circ) \to (G, \cdot)$ given by $\phi(g) = g\psi(g^{-1})$.

ϕ is an isomorphism if and only if $\psi(g) \neq g$ for all $g \neq 1_G$.

This ϕ, which will always be implicitly dependent on ψ, will be important throughout this talk.

Abelian maps and brace blocks

A brace block is a set G together with a family of binary operations $\{\circ_i : i \in \mathcal{I}\}$ (where $\mathcal{I}$ is some index set) such that $(G, \circ_i, \circ_j)$ is a (biskew) brace for all $i, j \in \mathcal{I}$.

Let $\psi \in \mathrm{Ab}(G)$, and for each $n \geq 0$ define $\psi_n : G \to G$ by $\psi_n(g) = \phi^n(g)^{-1}g$.

Note $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$.

Generally, $\psi_n \in \mathrm{Ab}(G)$, allowing us to create a group $(G, \circ_n)$ with

$$g \circ_n h = g\psi_n(g^{-1})h\psi_n(g).$$

Can show: $(G, \{\circ_n : n \geq 0\})$ is a brace block.

Abelian maps and bracoids

We can also get non-brace bracoids from an abelian map.

Let $\psi \in \mathrm{Ab}(G)$ and construct the brace as above.

Recall $\phi : (G, \circ) \to (G, \cdot),\ \phi(g) = g\psi(g^{-1})$ is a homomorphism.

We say $H \leq G$ is ψ-admissible if $[G, \phi(H)] \subseteq H$.

Suppose $H \leq G$ is ψ-admissible. Let $N = G/H$ (left cosets (with respect to $\cdot$)), then

$$xH \star yH = (x \circ y)H = x\psi(x^{-1})y\psi(x)H,\ xH, yH \in N$$

makes $(N, \star)$ into a group.

Letting $g \odot xH = (gx)H$ makes (G, N) a bracoid.

Furthermore, this construction works if and only if H is ψ-admissible.

A construction of Martin-Lyons and T.

One way to construct bracoids is the following:

Let $\mathfrak{B} = (B, \circ, \cdot)$ be a brace.

Let A be a strong left ideal of $\mathfrak{B}$–that is:

- $A \trianglelefteq (B, \circ)$;
- $A \leq (B, \cdot)$;
- $b^{-1} \circ (b \cdot a) \in A$ for all $a \in A,\ b \in B$.

Then $(B, \cdot, B/A, \star, \odot)$ is a bracoid, where $bA \star cA = (b \circ c)A$ and $b \odot cA = bcA$.

Fact. If $\psi \in \mathrm{Ab}(G)$ and H is ψ-admissible then H is a strong left ideal of $(G, \circ, \cdot)$, and our bracoid construction can be seen as an example of this more general case.

$$[G, \phi(H)] \subseteq H, \ \phi(g) = g\psi(g^{-1})$$

Example

Let $G = S_n, \ n \geq 3$. Define $\psi : G \to G$ by

$$\psi(\sigma) = \begin{cases} 1_G & \sigma \in A_n \\ (12) & \sigma \notin A_n \end{cases}.$$

Since $\psi(G) = \langle (12) \rangle$ this is an abelian map.
Let $H = \langle (12) \rangle$. Then $\phi(H) = \{1_G\}$ and $[G, \{1_G\}] \subseteq H$.
Hence H is ψ-admissible.
Write $N = G/H$. The group $(N, \star) \cong A_n$.
If $n = 3$ then G acts on N as in the previous example.

Do examples exist?

Question

Does every abelian map yield ψ-admissible subgroups?

Yes:

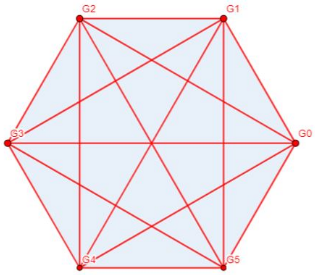
- $H = G$, $H = \{1_G\}$, though these are not terribly interesting.
- $H = \ker\psi$ since $\phi(k) = k\psi(k^{-1}) = k$ for $k \in \ker\psi$, and $[G, \ker\psi] \subseteq \ker\psi$ since $\ker\psi \trianglelefteq G$.
- $H = \operatorname{fix}\psi$, the subgroup of fixed points, since $\phi(H) = \{1_G\}$.
  But $\operatorname{fix}\psi$ may be trivial.
- $H \leq \operatorname{fix}\psi$.
- $H = \operatorname{fix}\psi \ker\psi$ (which could possibly be all of G).

However, most subgroups tend not to be ψ-admissible.

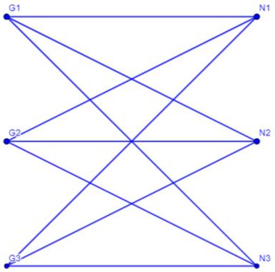
Outline

Definition

A bracoid web is a collection $(G, \{\circ_m : m \geq 0\}, N, \{\star_n : n \geq 0\}, \{\odot_{m,n} : m, n \geq 0\})$ such that $(G, \circ_m, N, \star_n, \odot_{m,n})$ is a bracoid for every $m, n \geq 0$.



Brace Block: Complete Graph



Bracoid Web: Bipartite Graph

Abelian maps and bracoid webs

Let $\psi \in \mathrm{Ab}(G)$, and recall $\phi(g) = g\psi(g^{-1})$.

We say $H \leq G$ is fully ψ-admissible if $[G, \phi^n(H)] \subseteq H$ for all $n \geq 1$.

Examples

1. $H = \mathrm{fix}\,\psi$. Then $[G, \phi^n(H)] = [G, \{1_G\}] = \{1_G\} \subseteq H$.
2. $H = \ker\psi$. Then $[G, \phi^n(H)] = [G, H] \subseteq H$ since $H \trianglelefteq G$.
3. $H = \mathrm{fix}\,\psi \ker\psi$. Then $[G, \phi^n(H)] = [G, \ker\psi] \subseteq H$.

$[G, \phi^n(H)] \subseteq H.$

Key concepts.

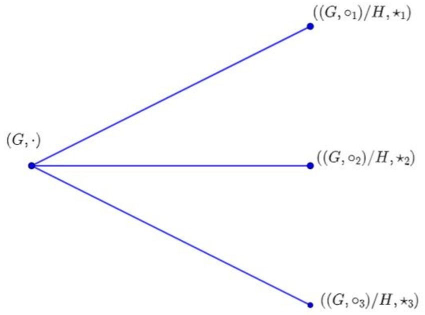
- If $\psi \in \mathrm{Ab}(G)$, so is $\psi_n : g \mapsto \phi^n(g)^{-1}g$ for all n, allowing us to form the group $(G, \circ_n)$.
- A subgroup H is ψ_n-admissible iff $[G, \phi_n(H)] \subseteq H$, where $\phi_n(h) = h\psi_n(h)^{-1} = h(h^{-1}\phi^n(h)) = \phi^n(h)$.
- A subgroup H is ψ_n-admissible iff $[G, \phi^n(H)] \subseteq H$.

Let $N = G/H$ (left cosets) and define $xH \star_n yH = (x \circ_n y)H$. Then $(G, \cdot, N, \star_n, \odot)$ is a bracoid, where $g \odot xH = gxH$.

First result

Theorem

Let $H \leq (G, \cdot)$ be fully ψ-admissible. Then $(G, \cdot, N, \{\star_n : n \geq 1\}, \odot)$ is a bracoid web.



First Bracoid Web

Casting a larger web

Of course, given $\psi \in \mathrm{Ab}(G)$ we have a collection of groups $\{(G, \circ_m) : m \geq 0\}$ with $a \circ_0 b = a \cdot b$.

Let $[a, b]_m$ be the commutator of a and b in $(G, \circ_m)$.

Proposition

If $H \leq (G, \cdot)$ is fully ψ-admissible then $[G, \phi^n(H)]_m \subseteq H$ for all $m, n \geq 1$, and $H \leq (G, \circ_m)$.

So $(G, \circ_m, N, \star_n, \odot_{m,n})$ is a brace as well, where $N = (G, \circ_n)/H$ and $g \odot_m (x \circ_n H) = (g \circ_m x) \circ_n H$.

On cosets

For $\psi \in \mathrm{Ab}(G)$ and H a ψ-admissible subgroup, there appear to be two types of cosets:

$$xH = \{xh : h \in H\} \quad x \circ H = \{x \circ h : h \in H\}.$$

$$\begin{aligned} x \circ h &= x\psi(x^{-1})h\psi(x) \\ &= x\psi(x^{-1})h\psi(h^{-1}xh) \\ &= x\psi(x^{-1})\phi(h)\psi(x)\phi(h)^{-1}h \\ &= x[\psi(x^{-1}), \phi(h)]h \\ &= xh',\ h' \in H. \end{aligned}$$

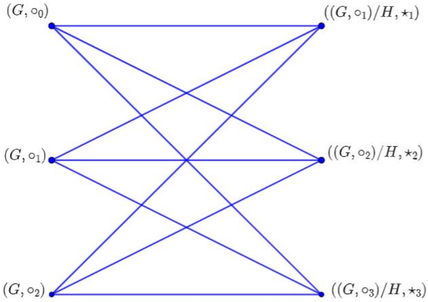
Thus the cosets coincide. Generally, $x \circ_n H = xH$.

For context, we will continue to represent the cosets as $x \circ_n H$ when $N = (G, \circ_n)/H$.

Main result

Theorem

Let $H \leq (G, \cdot)$ be fully ψ-admissible. Then $(G, \{\circ_m : m \geq 0\}, \cdot, N, \{\star_n : n \geq 1\}, \{\odot_{m,n} : m \geq 0, n \geq 1\})$ is a bracoid web.



Bracoid Web

Outline

Recall $\psi_m(g) = \phi^m(g)^{-1}g$

Let $G = F(a, b)$ be a free group.

Let $\psi : G \to G$ be given by $\psi(a) = 1_G,\ \psi(b) = b^{-1}$.

Generally, $\psi\left(\prod a^{r_i} b^{s_i}\right) = b^{s(g)}$ where $s(g) = \sum s_i$.

Then $\psi(G) = \langle b \rangle$ and $\psi \in \mathrm{Ab}(G)$.

Note that $\phi(a^i) = a^i,\ \phi(b^i) = b^{2i}$.

Then $\psi_m(a) = 1$ and $\psi_m(b) = b^{1-2^m}$ for all $m \geq 1$.

For $m \geq 0$ operation in $(G, \circ_m)$ is

$$g \circ_m h = g\psi_m(g^{-1})h\psi_m(g) = gb^{(2^m-1)s(g)}hb^{(1-2^m)s(g)}.$$

Note that all of the binary operations are different because $\psi_{m_1} \neq \psi_{m_2},\ m_1 \neq m_2$ and $Z(F(a, b))$ is trivial.

Also, $(G, \circ_m)$ nonabelian since $a \circ_m b = ab$ and $b \circ_m a = b^{2^m}ab^{1-2^m}$.

Let $H = \langle b^2 \rangle = \mathrm{fix}\,\psi$

So H is fully ψ-admissible. Let us try to understand $N = (G, \circ_n)/H$.

We have $x \circ_n H = y \circ_n H$ iff $\psi^n(y)y^{-1}\psi^n(y^{-1}) \circ_n x \in \langle b^2 \rangle$.

Let $\overline{y} = \psi^n(y)y^{-1}\psi^n(y^{-1})$ (inverse to $y \in (G, \circ_n)$).

Since

$$(a \circ_n H) \star_n (b \circ_n H) = (ab) \circ_n H$$

$$(b \circ_n H) \star_n (a \circ_n H) = (b^{2^n} ab^{1-2^n}) \circ_n H$$

$$\overline{ab} \circ_n b^{2^n} ab^{1-2^n} \notin \langle b^2 \rangle$$

$$\overline{b^{2^n} ab^{-2^n}} \circ_n (b^{2^{n'}} ab^{-2^{n'}}) \notin \langle b^2 \rangle, \; n \neq n'$$

the groups $(G, \star_n)$ are nonabelian and pairwise distinct.

Let $g \odot_{m,n} (x \star_n H) = (g \circ_m x) \star_n H$. The resulting bracoid web is $(G, \{\circ_m : m \geq 0\}, G/H, \{\star_n : n \geq 1\}, \{\odot_{m,n} : m \geq 0\})$.

Outline

Greither-Pareigis Theory

Throughout this section, all groups and field extensions are finite.

Let:

- L/K be a separable field extension;
- E/L be an extension such that E/K is Galois;
- $G = \mathrm{Gal}(E/K),\ H = \mathrm{Gal}(E/L)$.

It follows from Greither-Pareigis theory that Hopf-Galois structures on L/K correspond to regular subgroups $N \leq \mathrm{Perm}(G/H)$ which are stabilized by G, where G acts on $\mathrm{Perm}(G/H)$ by conjugation by $\lambda(G)$ (left translation of cosets).

The isomorphism class of N is the type of the Hopf-Galois structure.

For $k \in G$ and $\eta \in N$ we write ${}^k\eta = \lambda(k)\eta\lambda(k^{-1})$.

We will explicitly describe two regular subgroups for each bracoid.

The First Hopf-Galois Structure

Let $\psi \in \mathrm{Ab}(G)$, and let $H \leq G$ be ψ-admissible.

Define $N = \{\eta_g : g \in G\} \subseteq \mathrm{Perm}(G/H)$ where

$$\eta_g[xH] = (g \circ x)H = gH \star xH.$$

Then N is a subgroup of $\mathrm{Perm}(G/H)$: $\eta_{g_1}\eta_{g_2} = \eta_{g_1 \circ g_2}$.

Fact. $\eta_{g_1} = \eta_{g_2}$ if and only if $g_1 g_2^{-1} \in H$, so $|N| = |G/H|$.

Since $\eta_g[1_G H] = gH$ the subgroup $N \leq \mathrm{Perm}(G/H)$ is transitive, hence regular.

Can show ${}^k\eta_g = \eta_{k(g \circ k^{-1})}$ hence the action is G-stable.

So $N \leq \mathrm{Perm}(G/H)$ gives a Hopf-Galois structure on L/K.

The Second HGS

Keep the notation from above.

Define $P = \{\pi_g : g \in G\} \leq \mathrm{Perm}(G/H)$ where

$$\pi_g[xH] = (x \circ g)H = xH \star gH.$$

As before:

- P is indeed a subgroup; here $\pi_{g_1}\pi_{g_2} = \pi_{g_2 \circ g_1}$.
- $\pi_{g_1} = \pi_{g_2}$ iff g_1 and g_2 are in the same left coset;
- P is regular;
- P is G-stable: ${}^k\pi_g = \pi_{k(k^{-1} \circ g)}$.

So $P \leq \mathrm{Perm}(G/H)$ gives a Hopf-Galois structure on L/K, distinct from N if $(G, \circ)$ is nonabelian.

But it's a bit messy

Let $H \leq G$ be fully ψ-admissible.

For $m \geq 0$, let E_m/K_m be a Galois extension, $\mathrm{Gal}(E_m/K_m) = (G, \circ_m)$.

Let $N_n = \{\eta_g^{(m,n)} : g \in (G, \circ_m)\} \leq \mathrm{Perm}((G, \circ_m)/H)$ with $\eta_g^{(m,n)}[x \circ_m H] = (g \circ_n x) \circ_m H$.

This gives a Hopf-Galois structure on $L_{m,n} := E_m^{N_n}$.

Similar results hold for P_n.

So we may get Hopf-Galois structures on field extensions we (presumably) don't care about.

Another example

Let $G = D_n \times D_n = \langle r, s, t, u \rangle$, $|r| = |t| = n$, $|s| = |u| = 2$.

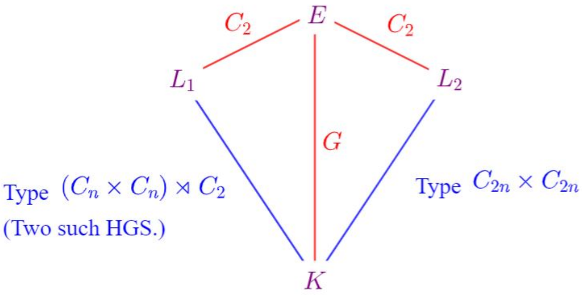
Define $\psi \in \mathrm{Ab}(G)$ by $\psi(r) = \psi(t) = 1_G$, $\psi(s) = u$, $\psi(u) = s$.

Then $H = \langle su \rangle$ is fully ψ-admissible.

Can compute the entire bracoid web:

$$\begin{array}{ll} (G, \circ_0) = D_n \times D_n & \\ (G, \circ_1) \cong C_2 \times ((C_n \times C_n) \rtimes C_2) & (N, \star_1) \cong (C_n \times C_n) \rtimes C_2 \\ (G, \circ_2) \cong C_{2n} \times C_{2n} & (N, \star_2) \cong C_{2n} \\ (G, \circ_n) = (G, \circ_2) & (N, \star_n) = (N, \star_2), \ n \geq 2. \end{array}$$

So we get Hopf-Galois structures on two subextensions of three different Galois extensions...



$$G \cong D_n \times D_n,\ C_2 \times ((C_n \times C_n) \rtimes C_2),\ \text{or}\ C_{2n} \times C_{2n}$$

Classical Galois Structures

Regular subgroups of $\mathrm{Perm}(G/C_2)$

Outline

Unresolved issues

- Does "ψ-admissible" imply "fully ψ-admissible"?
  We have no example of $H \le G$ such that $[G, \phi(H)] \le H$ but $[G, \phi^n(H)] \not\le H$ for some n.
  We expect they exist.
- Under what conditions are the bracoids in a brace web reduced?
  Recall reduced means that no nontrivial element of $(G, \circ_m)$ acts trivially on $(N, \star_n)$.

Unresolved issues

- What is the group structure of $(G/H, \star_n)$?
  Can be difficult in general.
  If $H = \text{fix}\,\psi$, then $H = \ker\phi$, hence

  $$(G/H, \star_1) = (G, \circ)/H \cong \phi(G) \leq (G, \cdot)$$

  hence $(G/H, \star_1)$ can be realized as a subgroup of G.
  In general, $H \neq \text{fix}\,\psi_n$.
- Can this construction be extended to recent generalizations of "abelian" maps (Caranti-Stefanello, K., Stefanello-Trappeniers)?
  Possible, but it seems the definition of ψ-admissible would need to change.

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Thank you.