# Normalizing Graphs of Regular Permutation Groups

Timothy Kohl

Boston University

May 30, 2022

# Hopf-Galois Structures

For $L/K$ a Galois extension with $G = Gal(L/K)$, a Hopf-Galois structure on $L/K$ corresponds to a regular subgroup $N \leq B = Perm(G)$ where the Hopf algebra which acts is $H = (L[N])^G$, where $G$ acts on $L$ via the Galois action, and on $N$ via the left regular representation $\lambda(G)$, under the condition that $\lambda(G) \leq Norm_B(N)$, i.e. $\lambda(G)$ normalizes $N$.

Recall that $N \leq B$ is regular if any two of the following conditions hold:

- $N$ acts transitively
- $N$ acts without fixed points (i.e. $\eta(x) = x$ for some $x \in G$ only if $\eta$ is the identity)
- $|N| = |G|$.

The regularity condition does not require that $N \cong G$, so one can subdivide those $N$ which give rise to Hopf-Galois structures by isomorphism class, so we define:

- $R(G) = \{N \leq B \mid \lambda(G) \leq Norm_B(N)$ where $N$ is regular$\}$
- $R(G,[M]) = \{N \in R(G) \mid N \cong M\}$

where $[M]$ is an isomorphism class of group of order $|G|$.

As such, each distinct Hopf-Galois structure corresponds to an element $N \in R(G,[M])$.

# Skew Braces and Bi-Skew Braces

## Definition

A skew left brace is a finite set $G$ together with two operations $\star$ and $\circ$ such that $(G,\star)$ and $(G,\circ)$ are both groups, where the two group operations satisfy the 'brace relation'

$$a \circ (b \star c) = (a \circ b) \star a^{-1} \star (a \circ c)$$

where '$a^{-1}$' is the inverse of $a$ in $(G,\star)$.
And we denote this by $(G,\circ,\star)$.

There is a known relationship between skew-left braces $(G, \circ, \star)$ and Hopf-Galois structures.

The structure $(G, \circ)$ may be embedded into $B = Perm(G)$ by the left regular representation $\lambda_\circ : (G, \circ) \to B$.

One finds then that for $N = (G, \star)$, $N$ is embedded in $B$ as well, where $N$ is normalized by $\lambda_\circ(G)$.

Conversely, for $N \in R(G, [M])$ the group $N$ (being a regular subgroup of $B = Perm(G)$) gives rise to a skew left brace $(G, \circ, \star)$ where (by slight abuse of notation) $G = (G, \circ)$ and $N = (G, \star)$.

A bi-skew brace is defined to be a set $G$ together with two group operations $(G, \circ)$ and $(G, \star)$ where the skew-brace relations hold symmetrically, that is:

$$a \circ (b \star c) = (a \circ b) \star a^{-1} \star (a \circ c)$$
$$a \star (b \circ c) = (a \star b) \circ \bar{a} \circ (a \star c)$$

where '$\bar{a}$' is the inverse of $a$ in $(G, \circ)$.

This means that $(G, \circ, \star)$ and $(G, \star, \circ)$ are skew braces.

In terms of regular subgroups embedded in $Perm(G)$, we still have two groups $(G,\circ)$ embedded as $\lambda_{\circ}(G)$ and $N=(G,\star)$.

The $(G,\circ,\star)$ structure frames $N$ as being normalized by $\lambda_{\circ}(G)=\lambda(G)$ while the complementary brace structure $(G,\star,\circ)$ means that $N$ normalizes $\lambda(G)$.

For $G$ embedded in $B=Perm(G)$ as $\lambda(G)$, recall that the holomorph of $G$ is

$$Norm_B(\lambda(G))$$

, so $(G,\circ,\star)$ and $(G,\star,\circ)$ being braces simultaneously means that $N$ is actually to be found as a regular subgroup of $\mathrm{Hol}(G)$.

The cases we wish to explore are those bi-skew braces $(G, \circ, \star)$ / $(G, \star, \circ)$ where $(G, \circ) \cong N \cong (G, \star)$.

As such, we are exploring regular subgroups of $\mathrm{Hol}(G)$ normalized by $\lambda(G)$.

We are also working into this discussion, particular certain subsets of these which generalize the notion of the multiple holomorph.

As such, some of the source material for this is [4], which we are extrapolating to the study of bi-skew braces.

For $B = Perm(G)$ we define

$$\mathcal{R}(G) = \{N \leq B \mid N \text{ is regular, } N \cong G,\ \lambda(G) \leq Norm_B(N)\}$$
$$\mathcal{S}(G) = \{M \leq Hol(G) \mid M \text{ is regular and } M \cong G\}$$

so the set

$$\mathcal{S}(G) \cap \mathcal{R}(G)$$

consists of those regular subgroups $N \leq Perm(G)$ (where $N \cong G$) with the property that they normalize, and are normalized by, $\lambda(G)$.

We observe an important symmetry between $\mathcal{S}(G)$ and $\mathcal{R}(G)$.

The fact that these $N$ are regular, and isomorphic to $\lambda(G)$ implies that each such $N$ is a conjugate of $\lambda(G)$ by some element of $B$.

As such, one can show that

$$\gamma\lambda(G)\gamma^{-1} \in \mathcal{S}(G) \leftrightarrow \gamma^{-1}\lambda(G)\gamma \in \mathcal{R}(G)$$

and given that $\lambda(G) \in \mathcal{S}(G) \cap \mathcal{R}(G)$ obviously, we observe that $id_B$ certainly conjugates $\lambda(G)$ to itself.

As such, one can show (by a theorem due to König) that there exists a set

$$\pi(\mathcal{S}(G) \cap \mathcal{R}(G)) = \{\beta_1, \ldots, \beta_t\}$$

(which we can assume contains $id_B$) of distinct coset representatives of $\mathrm{Hol}(G)$ with the property that

$$\mathcal{S}(G) \cap \mathcal{R}(G) = \{\beta_i \lambda(G) \beta_i^{-1}\}$$

where if $\beta\lambda(G)\beta^{-1} \in \mathcal{S}(G) \cap \mathcal{R}(G)$ then $\beta^{-1}\lambda(G)\beta \in \mathcal{S}(G) \cap \mathcal{R}(G)$.

Note, this does not mean that $\pi(\mathcal{S}(G) \cap \mathcal{R}(G))$ is itself closed under inverses.

Any $\pi(\mathcal{S}(G) \cap \mathcal{R}(G)) = \{\beta_1, \ldots, \beta_t\}$ gives rise to a set of distinct cosets $\beta_i Hol(G)$.

A natural question to ask is whether these cosets (or even the coset representatives) form a group?

Before answering this, we first consider a subset of $\mathcal{S}(G) \cap \mathcal{R}(G)$ for which the answer is more easily understood, and whose properties we are trying to generalize.

## Definition

For $G$ a group, embedded in $B = Perm(G)$ as $\lambda(G)$, define

$$\begin{aligned}\mathcal{H}(G) &= \{N \leq B \mid N \text{ regular },\ N \cong G,\ Norm_B(N) = \mathrm{Hol}(G)\}\\ &= \{N \triangleleft \mathrm{Hol}(G) \mid N \text{ regular },\ N \cong G\}\end{aligned}$$

It is clear therefore that $\mathcal{H}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$.

And since $\mathcal{H}(G)$ consists of regular subgroups isomorphic to $\lambda(G)$ then they are conjugate to $\lambda(G)$, and so $\mathcal{H}(G)$ determines a set of cosets $\{\gamma_i \operatorname{Hol}(G)\}$ of $\operatorname{Hol}(G)$.

But for any such $\gamma$ we have that

$$\gamma \operatorname{Norm}_{\mathrm{B}}(\lambda(G))\gamma^{-1} = \operatorname{Norm}_{\mathrm{B}}(\gamma\lambda(G)\gamma^{-1}) = \operatorname{Norm}_{\mathrm{B}}(\lambda(G))$$

and so $\gamma$ normalizes $\operatorname{Hol}(G)$, that is, it's an element of the so-called *multiple holomorph* $NHol(G) = \operatorname{Norm}_{\mathrm{B}}(\operatorname{Hol}(G))$.

As such $\mathcal{H}(G)$ is the orbit of $\lambda(G)$ under the action of $NHol(G)$.

And the cosets which correspond to the elements of $\mathcal{H}(G)$ give rise to a group, which contains $\mathrm{Hol}(G)$ as a normal subgroup.

Moreover, there is an obvious action of $T(G) = NHol(G)/\mathrm{Hol}(G)$ on $\mathcal{H}(G)$ and indeed, $\mathcal{H}(G)$ is the orbit of $\lambda(G)$ under the action of $T(G)$, and this action is regular.

One of our goals was to come up with a similar group (containing $\mathrm{Hol}(G)$) which acts transitively on $\mathcal{S}(G) \cap \mathcal{R}(G)$, with $\mathrm{Hol}(G)$ as the stabilizer of $\lambda(G)$.

$NHol(G)$ is obviously the maximal subgroup of $B$ which contains $\mathrm{Hol}(G)$ as a normal subgroup.

So in particular, $NHol(G)$ is a group extension of $\mathrm{Hol}(G)$.

As such, if $\mathcal{S}(G) \cap \mathcal{R}(G)$ is properly larger than $\mathcal{H}(G)$ then the cosets of $\mathrm{Hol}(G)$ formed from the coset representatives

$$\pi(\mathcal{S}(G) \cap \mathcal{R}(G)) = \{\beta_1, \dots, \beta_t\}$$

do not come from a group with $\mathrm{Hol}(G)$ as a normal subgroup.

However, we can give conditions for when $\cup \beta_i \,\mathrm{Hol}(G)$ forms a group which contains $\mathrm{Hol}(G)$, but *not* as a normal subgroup.

Another feature of $\mathcal{H}(G)$ to note is that any pair $N_1, N_2 \in \mathcal{H}(G)$ normalize each other.

The reason for this is that $Norm_B(N_1) = Norm_B(N_2) = \mathrm{Hol}(G)$.

Looking at the totality of $\mathcal{S}(G) \cap \mathcal{R}(G)$, there is no reason to expect its members to mutually normalize each other, but we can give some criteria which imply when this occurs.

## Definition

A set $P \subseteq B = Perm(G)$, of coset representatives of $\mathrm{Hol}(G)$ is 'conj-closed' if for each $\alpha, \beta \in P$, there exists a $\gamma \in P$ such that $\alpha\beta\lambda(G)\beta^{-1}\alpha^{-1} = \gamma\lambda(G)\gamma^{-1}$.

If so then we can define a binary operation $*$ in $P$ by $\alpha * \beta = \gamma$.

And we say $P$ is 'inv-closed' if for each $\beta \in P$, there exists and $\alpha \in P$ such that $\beta^{-1}\lambda(G)\beta = \alpha\lambda(G)\alpha^{-1}$.

The resemblance of these two properties to group laws is deliberate, and indeed, if $P$ conj-closed then $(P, *)$ is a (left) quasigroup, and if we assume that $P$ contains $id_B$, then $(P, *)$ is what is known as a (left) loop.

The relevance to mutual normalization comes from this.

## Proposition

*If $\mathcal{X}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$ where $\pi(\mathcal{X}(G)) = \{\beta_1, \ldots, \beta_m\}$ are the elements of $B$ which conjugate $\lambda(G)$ to each element of $\mathcal{X}(G)$, and $\pi(\mathcal{X}(G))$ is conj-closed, then all $N \in \mathcal{X}(G)$ normalize each other.*

As such, if the elements of $\mathcal{X}(G)$ *fail* to normalize each other, then it would contradict the conj-closed property.

The converse of this is not generally true, but we can say this.

**Proposition**

*Let $\pi(\mathcal{S}(G)\cap\mathcal{R}(G))$ parameterize $\mathcal{S}(G)\cap\mathcal{R}(G)$. If the elements of $\mathcal{S}(G)\cap\mathcal{R}(G)$ are mutually normalizing, and if $\pi(\mathcal{S}(G)\cap\mathcal{R}(G))$ is inv-closed then $\pi(\mathcal{S}(G)\cap\mathcal{R}(G))$ is conj-closed.*

And as mentioned above, there is always a $\pi(\mathcal{S}(G)\cap\mathcal{R}(G))$ which is inv-closed.

The one issue to confront is that it is not always the case that all the members of $\mathcal{S}(G)\cap\mathcal{R}(G)$ are mutually normalizing.

## Proposition

*Let $\mathcal{X}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$ (containing $\lambda(G)$) be parameterized by $\pi(\mathcal{X}(G)) = \{\beta_1, \ldots, \beta_m\}$, where $\pi(\mathcal{X}(G))$ is conj-closed and let $\mathrm{XHol}(G) = \cup_{i=1}^{m} \beta_i \mathrm{Hol}(G)$.*

*$\mathrm{XHol}(G)$ is a group, and the following properties hold:*

(a) *All the members of $\mathcal{X}(G)$ normalize each other.*

(b) $Orb_{\mathrm{XHol}(G)}(\lambda(G)) = \mathcal{X}(G)$

(c) $|\mathrm{XHol}(G)| = |\mathcal{X}(G)| \cdot |\mathrm{Hol}(G)|$

For example, $\mathcal{X}(G) = \mathcal{H}(G)$ (whence $\mathrm{XHol}(G) = NHol(G)$) certainly satisfies the conditions above, but again, we would like to find a larger subset of $\mathcal{S}(G) \cap \mathcal{R}(G)$ for which this is true, and for which there is a group which acts transitively.

Changing notation slightly, and with the view of the set of coset representatives/parameters forming a 'Q'uasi-group, we define the following subset of $\mathcal{S}(G) \cap \mathcal{R}(G)$.

**Definition**

Let $\mathcal{Q}(G) = \bigcap_{N \in \mathcal{S}(G) \cap \mathcal{R}(G)} \{M \in \mathcal{S}(G) \cap \mathcal{R}(G) \mid N \text{ normalizes } M\}$.

We first observe the most important property of this set.

**Lemma**

*The members of $\mathcal{Q}(G)$ mutually normalize each other.*

And as our goal is to generalize $\mathcal{H}(G)$, we observe.

## Lemma

*For $\mathcal{Q}(G)$ defined above, one has $\mathcal{H}(G) \subseteq \mathcal{Q}(G)$.*

## Proof.

If $M \in \mathcal{H}(G)$ then $\mathrm{Norm}_B(M) = \mathrm{Norm}_B(\lambda(G)) = \mathrm{Hol}(G)$, so for any $N \in \mathcal{S}(G) \cap \mathcal{R}(G)$ one has $N \leq \mathrm{Hol}(G) = \mathrm{Norm}_B(M)$. □

If $N \in \mathcal{Q}(G)$ then one can define $\mathcal{Q}(N)$ in the same way as $\mathcal{Q}(G)$ is defined, and the mutually normalizing property gives rise to an important symmetry.

**Proposition**

*If there exists a conj-closed $\pi(\mathcal{Q}(G))$ then for any $N \in \mathcal{Q}(G)$ one has that $\mathcal{Q}(N) = \mathcal{Q}(G)$.*

And if we define $N_1 \sim N_2$ if they have the same normalizer, then this is an equivalence relation and as a result we have a LaGrange type result.

**Proposition**

*For a given $G$ where there exists a conj-closed $\pi(\mathcal{Q}(G))$, if $N \in \mathcal{Q}(G)$ then $\mathcal{H}(N) \subseteq \mathcal{Q}(G)$ and so $|\mathcal{H}(G)| \big| |\mathcal{Q}(G)|$.*

Before going further, a bit of full disclosure must be made.

For any $G$, we have that

$$\mathcal{H}(G) \subseteq \mathcal{Q}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$$

and if $\mathcal{Q}(G) = \mathcal{H}(G)$ or $\mathcal{Q}(G) = \mathcal{S}(G) \cap \mathcal{R}(G)$, there does exist a $\pi(\mathcal{Q}(G))$ which is conj-closed, but it is not altogether obvious from the definition that it exists if these containments are proper.

However, there is a great deal of computational evidence that suggests that even when $\mathcal{H}(G) \subsetneq \mathcal{Q}(G) \subsetneq \mathcal{S}(G) \cap \mathcal{R}(G)$, there still exists a conj-closed $\pi(\mathcal{Q}(G))$.

The conj-closure property, being reminiscent of a group law is, again, not accidental, but is indeed the whole point, in that we wish to generalize the relationship between $\mathcal{H}(G)$ and $NHol(G)$ to the class $\mathcal{Q}(G)$, and this motivates the following definition:

**Definition**

For $\pi(\mathcal{Q}(G)) = \{\beta_1, \ldots, \beta_m\}$ which parameterizes $\mathcal{Q}(G)$, let the *quasiholomorph* of $G$ be $\mathrm{QHol}(G) = \cup_{i=1}^{m} \beta_i Hol(G)$.

And so, if there exists a conj-closed $\pi(\mathcal{Q}(G))$ then $\mathrm{QHol}(G)$ acts transitively on $\mathcal{Q}(G)$, paralleling how $NHol(G)$ acts transitively on $\mathcal{H}(G)$.

# QHol($G$) as a Zappa Szép Product

For QHol($G$), if there exists a $\pi(\mathcal{Q}(G))$ which is not only conj-closed, but is, in fact, a group in and of itself, then $\pi(\mathcal{Q}(G))$ would act regularly on $\mathcal{Q}(G)$.

In this case $\mathrm{QHol}(G) = \pi(\mathcal{Q}(G))\,\mathrm{Hol}(G)$ is a Zappa-Szép extension of Hol($G$) in that, unless $\mathcal{Q}(G) = \mathcal{H}(G)$, this is not a group extension of Hol($G$) in the usual sense.

Let's point out another reason for our interest in developing this, namely that $\mathcal{H}(G)$ is typically a relatively 'small' class of groups.

$T(G)$ and therefore $\mathcal{H}(G)$ has been computed for various classes of groups in [5], [6], [3], [1], [8].

There are instances when $\mathcal{H}(G)$ is exactly $\{\lambda(G), \rho(G)\}$ such as when $G$ is a simple group as shown in [2, Theorem 4].

Also, for abelian groups, Miller [5] showed that if $8 \not| |G|$ then

$$\mathcal{H}(G) = \{\lambda(G)\}$$

i.e. $T(G) = 1$.

However, we shall show that, for $G = C_{p^n}$ for example, $|\mathcal{Q}(G)| > 1$.

# Mutually Normalizing Subgroups of $Hol(C_{p^n})$

For odd primes $p$, we have the following.

**Proposition**

*For $G = C_{p^n}$ for $p$ odd, we have that $\mathcal{Q}(G) = \mathcal{S}(G) \cap \mathcal{R}(G)$ and $|\mathcal{Q}(G)| = p^{[\frac{n}{2}]}$.*

As such, for any conj-closed $\pi(\mathcal{S}(G) \cap \mathcal{R}(G)) = \pi(\mathcal{Q}(G))$ one has that QHol$(G)$ is a group, the question is whether it's a Zappa Szép product.

## Theorem

*If* $G = \langle\sigma\rangle$ *where* $\sigma = (1, 2, \ldots, p^n)$ *for* $p \geq 3$ *and* $n \geq 2$ *then* $\mathcal{Q}(G)$ *is parameterized by powers of the following element*

$$\beta = \prod_{i=1}^{p^{n-r}} \sigma_i^{t_{i-1}}$$

*where* $\sigma^{p^{n-r}} = \sigma_1\sigma_2\cdots\sigma_{p^{n-r}}$ *for* $r = [\frac{n}{2}]$ *and* $t_j = 1 + \frac{j(j+1)}{2}$.

## Corollary

*For* $G = C_{p^n}$*, with* $p$ *odd, the group* QHol$(G)$ *is a Zappa-Szép extension of* $Hol(G)$*, namely* $\pi(\mathcal{Q}(G))Hol(G)$ *where* $\pi(\mathcal{Q}(G)) = \langle\beta\rangle$.

# Cyclic 2-Groups

For $G = \langle\sigma\rangle \cong C_{2^n}$ the enumeration is nearly identical.

For $n = 1, 2$ the only cyclic subgroups of order $2^n$ of $Hol(G)$ are $G$ itself.

And in general, $|\mathcal{H}(G)| = 2$, and $\mathcal{S}(G) \cap \mathcal{R}(G) = \mathcal{Q}(G)$ so therefore any $\pi(\mathcal{Q}(G))$ is conj-closed and thus $\mathrm{QHol}(G)$ is a group.

As to whether $\mathrm{QHol}(G)$ is a Zappa-Szép extension of $\mathrm{Hol}(G)$, one may verify this for $n$ odd, and computational evidence implies that it is true for $n$ even as well.

# Cyclic groups in general

For general cyclic groups $C_n$ where $n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$ for distinct primes $p_i$ then $\mathrm{Hol}(C_n) \cong \mathrm{Hol}(C_{p_1^{e_1}}) \times \cdots \times \mathrm{Hol}(C_{p_r^{e_r}})$.

And of course, for groups $A$,$B$ of relatively prime order in general, $\mathrm{Hol}(A \times B) \cong \mathrm{Hol}(A) \times \mathrm{Hol}(B)$, and similarly for the multiple holomorph.

For $\mathcal{Q}(G)$ we can prove something similar.

**Proposition**

*If $gcd(|G_1|, |G_2|) = 1$ then $\mathrm{QHol}(G_1 \times G_2) \cong \mathrm{QHol}(G_1) \times \mathrm{QHol}(G_2)$.*

# Non-cyclic Groups

For cyclic groups $G$ we have that $\mathcal{Q}(G) = \mathcal{S}(G) \cap \mathcal{R}(G)$ but for non-cyclic groups, there are cases where $\mathcal{Q}(G)$ is a proper subset of $\mathcal{S}(G) \cap \mathcal{R}(G)$, which means that not all of the groups in $\mathcal{S}(G) \cap \mathcal{R}(G)$ have the mutually normalizing property.

For dihedral groups $D_n = \langle x, t \mid x^n = t^2 = 1, xt = tx^{-1} \rangle$ we have the following

- If $n$ is odd, then $|\mathcal{Q}(D_n)| = |\mathcal{H}(D_n)| = |\Upsilon_n|$.
- If $n$ is even $|\mathcal{Q}(D_n)| = \begin{Bmatrix} |\Upsilon_n| \text{ if } 8 \not| n \\ 2|\Upsilon_n| \text{ if } 8|n \end{Bmatrix}$

where $\Upsilon_n = \{u \in U_n \mid u^2 = 1\}$.

And so if $8 \not| n$, $\mathrm{QHol}(D_n) = \mathrm{NHol}(D_n)$ which is a split extension of $\mathrm{Hol}(D_n)$ by the group $M_n = \{\tau_u \mid u \in \Upsilon_n\}$ where $\tau_u(x^i) = x^i$ and $\tau_u(tx^i) = tx^{ui}$, which is an elementary abelian group.

If $8|n$ then there is an element $\mu$ of order 2 such that $M_n\langle\mu\rangle$ is a $\pi(\mathcal{Q}(D_n))$ where, $\mu$ commutes with those $\tau_u$ where $u \equiv 1$ $(mod\ 4)$.

For reference, when $n$ is odd, $\mathcal{H}(D_n) = \mathcal{Q}(D_n) = \mathcal{S}(D_n) \cap \mathcal{R}(D_n)$.

When $n$ is even it turns out that

$$|\mathcal{S}(D_n) \cap \mathcal{R}(D_n)| = \begin{Bmatrix} n + |\Upsilon_n| \text{ if } 8 \not| n \\ n + 2|\Upsilon_n| \text{ if } 8|n \end{Bmatrix}$$

which means that when $8|n$, we have proper containments

$$\mathcal{H}(D_n) \subsetneq \mathcal{Q}(D_n) \subsetneq \mathcal{S}(D_n) \cap \mathcal{R}(D_n)$$

The 'incidence matrix' indicates how members of $\mathcal{S}(D_8)\cap\mathcal{R}(D_8)$ normalize each other.

$\downarrow$ normalizes $\rightarrow$

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | * | * | * | * |
| | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | 3 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | * | * | * | * |
| $\lambda(D_8)$ | 4 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | 5 | * | 2 | * | 4 | 5 | 6 | * | 8 | * | 10 | 11 | 12 | * | * | * | * |
| $\mathcal{H}$ | 6 | * | 2 | * | 4 | 5 | 6 | * | 8 | * | 10 | 11 | 12 | * | * | * | * |
| | 7 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | * | * | * | * |
| $\mathcal{H}$ | 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | * | * | * | * |
| | 10 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $\rho(D_8)$ | 11 | * | 2 | * | 4 | 5 | 6 | * | 8 | * | 10 | 11 | 12 | * | * | * | * |
| | 12 | * | 2 | * | 4 | 5 | 6 | * | 8 | * | 10 | 11 | 12 | * | * | * | * |
| | 13 | * | 2 | * | 4 | 5 | 6 | * | 8 | * | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | 14 | * | 2 | * | 4 | 5 | 6 | * | 8 | * | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | 15 | * | 2 | * | 4 | 5 | 6 | * | 8 | * | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | 16 | * | 2 | * | 4 | 5 | 6 | * | 8 | * | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | $\mathcal{Q}$ | | 2 | | 4 | 5 | 6 | | 8 | | 10 | 11 | 12 | | | | |

So the members of $\mathcal{Q}$ are seen to be those $N$ in $\mathcal{S}\cap\mathcal{R}$ which appear in each column.

One other thing to note about QHol($G$) when $\pi(\mathcal{Q}(G))$ is a group itself, is that one can find ones in different isomorphism classes.

For example, with $G = C_4 \times C_4$, one has that $\mathcal{H}(G) = \{\lambda(G)\}$ but that $|\mathcal{Q}(G)| = |\mathcal{S}(G) \cap \mathcal{R}(G)| = 24$ and there are $\pi(\mathcal{Q}(G))$ in the following isomorphism classes:

$$C_4 \times S_3, (C_6 \times C_2) \rtimes C_2, C_3 \times D_8, S_4, C_2 \times A_4, C_2 \times C_2 \times S_3$$

And in the opposite direction, there are cases where $\mathrm{QHol}(G)$ is known to be a group, but that there are **no** $\pi(\mathcal{Q}(G))$ which are a group, so that $\mathrm{QHol}(G)$ is not a Zappa-Szép product, although $\mathrm{QHol}(G)$ acts transitively with $\mathrm{Hol}(G)$ being the stabilizer of $\lambda(G)$.

The lowest order example of this is for $G = (C_4 \times C_2) \rtimes C_2$.

In this case $\mathcal{H}(G) = \{\lambda(G), \rho(G)\}$ but $|\mathcal{Q}(G)| = |\mathcal{S}(G) \cap \mathcal{R}(G)| = 224$.

In fact, $\mathcal{S}(G) = \mathcal{R}(G) = \mathcal{S}(G) \cap \mathcal{R}(G) = \mathcal{Q}(G)$ so all Hopf-Galois structures of type $(G, [G])$ correspond to subgroups of the holomorph, and each pair gives rise to a bi-skew brace.

# Normalizing Graphs and Cliques

For a given group $G$, the set $\mathcal{S}(G) \cap \mathcal{R}(G)$, and the ways in which members of $\mathcal{S}(G) \cap \mathcal{R}(G)$ normalize (or fail to normalize) each other can be depicted by a 'normalizing graph'.

This is a graph where the nodes are the members of $\mathcal{S}(G) \cap \mathcal{R}(G)$ and an edge $N_1 \rightarrow N_2$ exists if $N_1$ normalizes $N_2$.

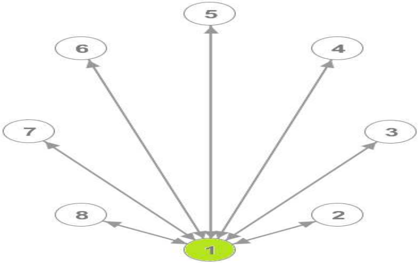
This graph is a directed graph generally, although if two members $N_1$ and $N_2$ normalize each other then one can view them as being connected by an edge, or a two headed arrow, $N_1 \leftrightarrow N_2$.

Within this graph, we can consider different subgraphs corresponding to subsets of $\mathcal{S}(G) \cap \mathcal{R}(G)$ such as $\mathcal{H}(G)$ and $\mathcal{Q}(G)$.

In particular, collections such as $\mathcal{Q}(G)$ and $\mathcal{H}(G)$ form complete sub-graphs or 'cliques' within $\mathcal{S}(G) \cap \mathcal{R}(G)$.

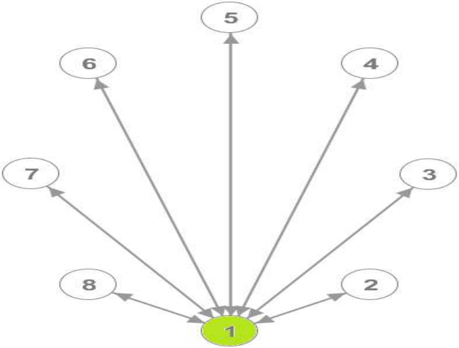
We have containments $\mathcal{H}(G) \subseteq \mathcal{Q}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$ where, for example, with $G$ cyclic as we've seen, $\mathcal{Q}(G) = \mathcal{S}(G) \cap \mathcal{R}(G)$ in that all members of $\mathcal{S}(G) \cap \mathcal{R}(G)$ are mutually normalizing.

And for some groups we have the opposite possibility, namely that $\mathcal{Q}(G) = \mathcal{H}(G) = \{\lambda(G)\}$ such as for the group $G = C_2 \times C_2 \times C_2$




where $\lambda(G) = 1$.

In this case, $\mathcal{S}(G) \cap \mathcal{R}(G)$ is a 'star graph' namely one with all nodes connected to a single center, namely $\lambda(G)$ itself.




But it's clear that one could form a bi-skew brace with $\lambda(G)$ and any other $N \in \mathcal{S}(G) \cap \mathcal{R}(G)$.

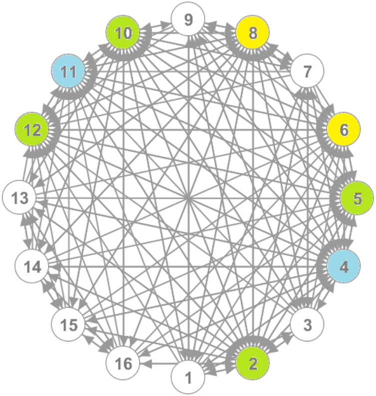
In general, it seems that $\mathcal{Q}(G) = \{\lambda(G)\}$ for elementary abelian groups.

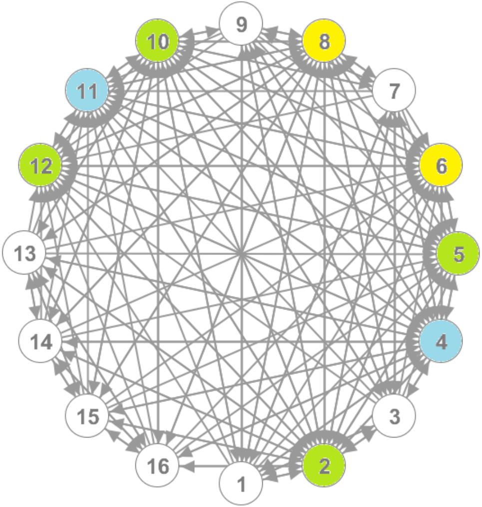
For example if $G = C_3 \times C_3 \times C_3$ then $\mathcal{S}(G) = \mathcal{R}(G)$ and $|\mathcal{R}(G)| = 339$ but $\mathcal{Q}(G) = \{\lambda(G)\}$.

For the case of $D_8$ we can visualize the fact that

$$\mathcal{H}(D_8) \subseteq \mathcal{Q}(D_8) \subseteq \mathcal{S}(D_8) \cap \mathcal{R}(D_8)$$

are all proper inclusions.





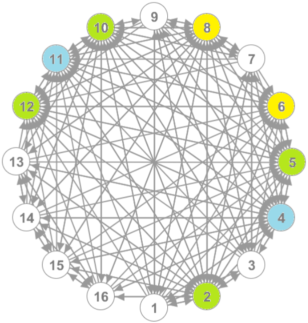

- $\mathcal{S}(D_8) \cap \mathcal{R}(D_8) = \{1, \ldots, 16\}$
- $\mathcal{Q}(D_8) = \{2, 4, 5, 6, 8, 10, 11, 12\}$
- $\mathcal{H}(D_8) = \{4, 6, 8, 11\}$
- $\lambda(D_8) = 4,\ \rho(D_8) = 11$

As observed earlier, the elements of $\mathcal{Q}(G)$ form a clique, namely a complete sub-graph of the normalizing graph of $\mathcal{S}(G) \cap \mathcal{R}(G)$, a natural question to ask is whether it is a *maximal* clique?

For $D_8$ one can show that $\mathcal{Q}(D_8)$ is the maximal clique which contains $\mathcal{H}(D_8)$.

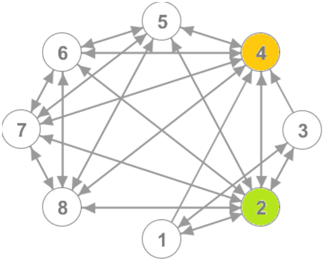
However, by 'brute force' calculation one can find other (maximal) cliques within $\mathcal{S}(D_8) \cap \mathcal{R}(D_8)$ but none contain $\mathcal{H}(D_8)$.

Indeed the other maximal cliques within $\mathcal{S}(D_8) \cap \mathcal{R}(D_8)$ are




[ 1, 2, 3, 4, 7, 8, 9, 10 ]
[ 2, 4, 8, 10, 13, 14, 15, 16 ]

For abelian groups $G$ such that $|T(G)| > 1$ we encounter some cases where $\mathcal{S}(G) \cap \mathcal{R}(G)$ contains maximal cliques $\mathcal{X}(G)$ which properly contain $\mathcal{Q}(G)$ where a given $\pi(\mathcal{X}(G))$ gives rise to a Zappa Szép product XHol$(G)$ which contains QHol$(G)$.




For example, let $G = C_4 \times C_2$ with graph.

- $\mathcal{S}(G) \cap \mathcal{R}(G) = \{1, \ldots, 8\}$
- $\mathcal{Q}(G) = \{2, 4\}$
- $\mathcal{H}(G) = \{2, 4\}$
- $\lambda(G) = 2$, $'\rho(G)' = 4$

Here $\mathcal{X}(G) = \{2, 4, 5, 6, 7, 8\}$ is the maximal clique, and it contains $\mathcal{Q}(G)$, and XHol$(G)$ is a group where the orbit of $\lambda(G)$ is $\mathcal{X}(G)$.

And even for cases like $G = C_3 \times C_3 \times C_3$, where $\mathcal{Q}(G) = \{\lambda(G)\}$ there are other cliques contained in $\mathcal{S}(G) \cap \mathcal{R}(G)$.

For example, there exists a clique in $\mathcal{X}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$ (containing $\mathcal{Q}(G) = \{\lambda(G)\}$) with 27 members, so the mutual normalizing property behaves quite differently (with regard to $\mathcal{Q}(G)$) when $G$ is abelian versus when it isn't.

And even though this is a clique, $\mathrm{XHol}(G)$ is not a group since many of the product of elements in $\pi(\mathcal{X}(G))$ conjugate $\lambda(G)$ to groups outside of $\mathcal{S}(G)$.

# Isomorphic Holomorphs and Bridged Cliques

We conclude by looking at collections of non-isomorphic groups $\{G_i\}$ of the same order with the property that they have isomorphic holomorphs.

The classical example of this is the case of $D_{2n}$ the dihedral group of order $4n$, and $Q_n$ the quaternionic or dicyclic group of order $4n$ for $n \geq 3$.

What this means operationally is that if we have $\lambda(G_1) \leq Hol(G_1) \leq B = Perm(G_1)$ then one can find a regular subgroup $G_2 \triangleleft \mathrm{Hol}(G_1)$ whence $Norm_B(G_2) = \mathrm{Hol}(G_1)$, where again $G_1 \not\cong G_2$.

Note, Mills [7] demonstrated that this is impossible for abelian groups, so in particular for any such pair of non-isomorphic groups $G_1$ and $G_2$ with isomorphic holomorphs, it must be the case that $|\mathcal{H}(G_i)| > 1$.

Beyond dihedral and quaternionic groups, there are others, and it's not just pairs of groups which can have this property.

For example, these four groups of order 48 have isomorphic holomorphs

$$
\begin{aligned}
&(C_3 \times D_4) \rtimes C_2 \\
&(C_3 \rtimes Q_2) \rtimes C_2 \\
&(C_3 \times Q_2) \rtimes C_2 \\
&C_3 \rtimes Q_4
\end{aligned}
$$

And as the order increases, there are even larger clusters of groups with this property.

If now we have a collection of non-isomorphic groups $\{G_i\}$ of the same order with isomorphic holomorphs and if we pick $G_1$ represented as $\lambda(G_1) \leq B = Perm(G_1)$ then one can find *within* $\mathrm{Hol}(G_1)$ regular normal subgroups isomorphic to the other $G_i$ with normalizer equal to $\mathrm{Hol}(G_1)$.

(Conjectural) If we then compute $\mathcal{Q}(G_1)$, and also $\mathcal{Q}(G_i)$ for the other $G_i$ it is likely that all $\mathcal{Q}(G_i)$ have the same size and that the union of all these $\mathcal{Q}(G_i)$ will have the mutually normalizing property (and therefore be a collection of bi-skew braces).

We have verified this computationally for the four groups of order 48 on the previous page.

Each $|\mathcal{Q}(G_i)| = 16$ and so we have $\binom{64}{2} = 2016$ distinct bi-skew braces.

Thank You!

# Appendix

We have the following tables which indicate the relative sizes for $\mathcal{S}(G) \cap \mathcal{R}(G)$, $\mathcal{Q}(G)$ and $\mathcal{H}(G)$ for different groups, as well as the isomorphism classes of $\pi(\mathcal{Q}(G))$ that may arise, when $\pi(\mathcal{Q}(G))$ is actually a group.

We include all groups of order at most 40, except for $n = 32$, excluding $D_n$ for $n$ odd, and most cyclic groups. But we do include the cyclic groups $C_4$, $C_{16}$ and also $C_{64}$ so as to address the 'opening' regarding whether $\mathrm{QHol}(C_{2^n})$ is a group for $n$ even. We also include $D_n$ for $n$ even.

We utilized GAP to compute these tables. As such, we adopt the notational conventions they use, in particular writing $D_{2n}$ for the $n$-th dihedral group for example.

| $G$ | $\lvert\mathcal{S}\cap\mathcal{R}\rvert$ | $\lvert\mathcal{Q}\rvert$ | $\lvert\mathcal{H}\rvert$ | $\pi(\mathcal{Q})$ |
|---|---|---|---|---|
| $C_4$ | 1 | 1 | 1 | 1 |
| $C_2 \times C_2$ | 1 | 1 | 1 | 1 |
| $S_3$ | 2 | 2 | 2 | $C_2$ |
| $C_4 \times C_2$ | 8 | 2 | 2 | $C_2$ |
| $D_8$ | 6 | 6 | 2 | $S_3,\ C_6$ |
| $Q_8$ | 2 | 2 | 2 | $C_2$ |
| $C_2 \times C_2 \times C_2$ | 8 | 1 | 1 | 1 |
| $C_3 \times C_3$ | 9 | 1 | 1 | 1 |
| $C_3 \rtimes C_4$ | 2 | 2 | 2 | $C_2$ |
| $A_4$ | 6 | 2 | 2 | $C_2$ |
| $D_{12}$ | 8 | 2 | 2 | $C_2$ |
| $C_6 \times C_2$ | 1 | 1 | 1 | 1 |
| $C_{16}$ | 4 | 4 | 2 | $C_4, C_2 \times C_2$ |
| $C_4 \times C_4$ | 24 | 24 | 1 | $C_4 \times S_3,\ (C_6 \times C_2) \rtimes C_2,\ C_3 \times D_8,\ S_4,\ C_2 \times A_4,\ C_2 \times C_2 \times S_3$ |
| $(C_4 \times C_2) \rtimes C_2$ | 76 | 4 | 4 | $C_2 \times C_2$ |
| $C_4 \rtimes C_4$ | 72 | 72 | 8 | $C_3 \times S_4,\ (C_3 \times A_4) \rtimes C_2$ |
| $C_8 \times C_2$ | 10 | 4 | 4 | $C_2 \times C_2$ |
| $C_8 \rtimes C_2$ | 10 | 4 | 4 | $C_2 \times C_2$ |
| $D_{16}$ | 16 | 8 | 4 | $C_4 \times C_2,\ D_8,\ C_2 \times C_2 \times C_2$ |
| $QD_{16}$ | 32 | 16 | 16 | $C_2 \times D_8$ |
| $Q_{16}$ | 16 | 8 | 4 | $C_4 \times C_2,\ D_8,\ C_2 \times C_2 \times C_2$ |
| $C_4 \times C_2 \times C_2$ | 146 | 1 | 1 | 1 |
| $C_2 \times D_8$ | 198 | 6 | 2 | $S_3,\ C_6$ |
| $C_2 \times Q_8$ | 66 | 2 | 2 | $C_2$ |
| $(C_4 \times C_2) \rtimes C_2$ | 224 | 224 | 2 | QHol($G$) is not a Zappa-Szép Extension of Hol($G$) |
| $C_2 \times C_2 \times C_2 \times C_2$ | 106 | 1 | 1 | 1 |
| $C_3 \times S_3$ | 7 | 2 | 2 | $C_2$ |
| $(C_3 \times C_3) \rtimes C_2$ | 38 | 2 | 2 | $C_2$ |
| $C_6 \times C_3$ | 9 | 1 | 1 | 1 |
| $C_5 \rtimes C_4$ | 2 | 2 | 2 | $C_2$ |
| $C_5 \rtimes C_4$ | 7 | 2 | 2 | $C_2$ |
| $D_{20}$ | 12 | 2 | 2 | $C_2$ |
| $C_{10} \times C_2$ | 1 | 1 | 1 | 1 |
| $C_7 \rtimes C_3$ | 9 | 2 | 2 | $C_2$ |
| $C_3 \rtimes C_8$ | 4 | 4 | 4 | $C_2 \times C_2$ |
| $SL(2,3)$ | 6 | 2 | 2 | $C_2$ |
| $C_3 \rtimes Q_8$ | 16 | 4 | 4 | $C_2 \times C_2$ |
| $C_4 \times S_3$ | 32 | 8 | 8 | $C_2 \times C_2 \times C_2$ |
| $D_{24}$ | 16 | 4 | 4 | $C_2 \times C_2$ |
| $C_2 \times (C_3 \rtimes C_4)$ | 20 | 4 | 4 | $C_2 \times C_2$ |

| $G$ | $\lvert\mathcal{S}\cap\mathcal{R}\rvert$ | $\lvert\mathcal{Q}\rvert$ | $\lvert\mathcal{H}\rvert$ | $\pi(\mathcal{Q})$ |
|---|---|---|---|---|
| $(C_6\times C_2)\rtimes C_2$ | 32 | 8 | 8 | $C_2\times C_2\times C_2$ |
| $C_{12}\times C_2$ | 8 | 2 | 2 | $C_2$ |
| $C_3\times D_8$ | 6 | 6 | 2 | $S_3,\ C_6$ |
| $C_3\times Q_8$ | 2 | 2 | 2 | $C_2$ |
| $S_4$ | 5 | 2 | 2 | $C_2$ |
| $C_2\times A_4$ | 9 | 2 | 2 | $C_2$ |
| $C_2\times C_2\times S_3$ | 44 | 2 | 2 | $C_2$ |
| $C_6\times C_2\times C_2$ | 8 | 1 | 1 | 1 |
| $C_5\times C_5$ | 25 | 1 | 1 | 1 |
| $C_9\times C_3$ | 33 | 3 | 1 | $C_3$ |
| $(C_3\times C_3)\rtimes C_3$ | 78 | 2 | 2 | $C_2$ |
| $C_9\rtimes C_3$ | 63 | 6 | 2 | $S_3,\ C_6$ |
| $C_3\times C_3\times C_3$ | 339 | 1 | 1 | 1 |
| $C_7\rtimes C_4$ | 2 | 2 | 2 | $C_2$ |
| $D_{28}$ | 16 | 2 | 2 | $C_2$ |
| $C_{14}\times C_2$ | 1 | 1 | 1 | 1 |
| $C_5\times S_3$ | 2 | 2 | 2 | $C_2$ |
| $C_3\times D_{10}$ | 2 | 2 | 2 | $C_2$ |
| $C_9\rtimes C_4$ | 2 | 2 | 2 | $C_2$ |
| $(C_2\times C_2)\rtimes C_9$ | 18 | 6 | 2 | $S_3,\ C_6$ |
| $D_{36}$ | 20 | 2 | 2 | $C_2$ |
| $C_{18}\times C_2$ | 3 | 3 | 1 | $C_3$ |
| $C_3\times(C_3\rtimes C_4)$ | 7 | 2 | 2 | $C_2$ |
| $(C_3\times C_3)\rtimes C_4$ | 38 | 2 | 2 | $C_2$ |
| $C_{12}\times C_3$ | 9 | 1 | 1 | 1 |
| $(C_3\times C_3)\rtimes C_4$ | 11 | 2 | 2 | $C_2$ |
| $S_3\times S_3$ | 55 | 4 | 2 | $C_4,\ C_2\times C_2$ |
| $C_3\times A_4$ | 42 | 6 | 2 | $S_3,\ C_6$ |
| $C_6\times S_3$ | 19 | 2 | 2 | $C_2$ |
| $C_2\times((C_3\times C_3)\rtimes C_2)$ | 56 | 2 | 2 | $C_2$ |
| $C_6\times C_6$ | 9 | 1 | 1 | 1 |
| $C_{13}\rtimes C_3$ | 15 | 2 | 2 | $C_2$ |
| $C_5\rtimes C_8$ | 4 | 4 | 4 | $C_2\times C_2$ |
| $C_5\rtimes C_8$ | 14 | 4 | 4 | QHol($G$) is not a Zappa-Szép Extension of Hol($G$) |
| $C_5\rtimes Q_8$ | 24 | 4 | 4 | $C_2\times C_2$ |
| $C_4\times D_{10}$ | 48 | 8 | 8 | $C_2\times C_2\times C_2$ |
| $D_{40}$ | 24 | 4 | 4 | $C_2\times C_2$ |
| $C_2\times(C_5\rtimes C_4)$ | 28 | 4 | 4 | $C_2\times C_2$ |
| $(C_{10}\times C_2)\rtimes C_2$ | 48 | 8 | 8 | $C_2\times C_2\times C_2$ |
| $C_{20}\times C_2$ | 8 | 2 | 2 | $C_2$ |

| $G$ | $\lvert\mathcal{S}\cap\mathcal{R}\rvert$ | $\lvert\mathcal{Q}\rvert$ | $\lvert\mathcal{H}\rvert$ | $\pi(\mathcal{Q})$ |
|---|---|---|---|---|
| $C_5 \times D_8$ | 6 | 6 | 2 | $S_3,\ C_6$ |
| $C_5 \times Q_8$ | 2 | 2 | 2 | $C_2$ |
| $C_2 \times (C_5 \rtimes C_4)$ | 24 | 4 | 4 | QHol($G$) is not a Zappa-Szép Extension of Hol($G$) |
| $C_2 \times C_2 \times D_{10}$ | 68 | 2 | 2 | $C_2$ |
| $C_{10} \times C_2 \times C_2$ | 8 | 1 | 1 | 1 |
| $C_{64}$ | 8 | 8 | 2 | $C_8,\ Q_8$ |

F. Dalla Volta A. Caranti.
The multiple holomorph of a finitely generated abelian group.
*J. Alg.*, 481:327–347, 2017.

Scott Carnahan and Lindsay Childs.
Counting Hopf Galois structures on non-abelian Galois field extensions.
*J. Algebra*, 218:81–92, 1999.

T. Kohl.
Multiple holomorphs of dihedral and quaternionic groups.
*Comm. Alg.*, 43:4290–4304, 2015.

T. Kohl.
Mutually normalizing regular permutation groups and zappa-szep extensions of the holomorph.
*Rocky Mountain J. Math.*, 52(2):567–598, 2022.

G.A. Miller.
On the multiple holomorphs of a group.
*Math. Ann.*, 66:133–142, 1908.

W.H. Mills.
Multiple holomorphs of finitely generated abelian groups.
*Trans. Amer. Math. Soc.*, 71:379–392, 1951.

W.H. Mills.
On the non-isomorphism of certain holomorphs.
*Trans. Amer. Math. Soc.*, 74:428–443, 1953.

C. Tsang.
On the multiple holomorph of a finite almost simple group.
*New York J. Math.*, 25:949–963, 2019.